SPIN-02/36

hep-th/0212239

On Lagrangians and Gaugings of Maximal Supergravities

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Abstract

A consistent gauging of maximal supergravity requires that the T-tensor transforms according to a specific representation of the duality group. The analysis of viable gaugings is thus amenable to group-theoretical analysis, which we explain and exploit for a large variety of gaugings. We discuss the subtleties in four spacetime dimensions, where the ungauged Lagrangians are not unique and encoded in an $\mathrm{E}_{7(7)}\backslash\mathrm{Sp}(56;\mathbb{R})/\mathrm{GL}(28)$ matrix. Here we define the T-tensor and derive all relevant identities in full generality. We present a large number of examples in $d = 4, 5$ spacetime dimensions which include non-semisimple gaugings of the type arising in (multiple) Scherk-Schwarz reductions. We also present some general background material on the latter as well as some group-theoretical results which are necessary for using computer algebra.

1 Introduction

Maximal supergravity theories contain a number of vector gauge fields which have an optional coupling to themselves as well as to other supergravity fields. The corresponding gauge groups are nonabelian. To preserve supersymmetry in the presence of these gauge couplings, the Lagrangian must contain masslike terms for the fermions and a potential depending on the scalar fields. In nonmaximal supergravity these terms are often described by means of auxiliary fields and/or moment maps; in the maximally supersymmetric theories the effect of the masslike terms and the potential is encoded in the so-called T-tensor [1]. It is a subtle matter to determine which gauge groups and corresponding charge assignments are compatible with supersymmetry. Based on Kaluza-Klein compactifications of higher-dimensional maximal supergravities on spheres, one readily concludes that the gauge groups SO(8), SO(6) and SO(5) are possible options in $d = 4, 5$ and 7 dimensions, respectively, corresponding to the isometry groups of S^7, S^5 and S^4 [1, 2, 3]. But also noncompact and non-semisimple groups turn out to be possible [4, 2, 5], which are noncompact versions and/or contractions of the orthogonal groups. More recent work revealed the so-called 'flat' gauge groups that one obtains upon Scherk-Schwarz reductions of higher-dimensional theories [6], as well as several other non-semisimple groups [7, 8, 9]. In $d = 3$ dimensions there is no guidance from Kaluza-Klein compactifications and one must rely on a group-theoretical analysis [10]. In this paper we apply the same kind of analysis to gaugings in higher dimensions.

Apart from the choice of the gauge group, a number of other subtleties arise that depend on the number of spacetime dimensions. In $d = 3$ dimensions supergravity does not contain any vector fields, because these can be dualized to scalar fields. Nevertheless a gauging can be performed by introducing vector fields via a Chern-Simons term (so that new dynamic degrees of freedom are avoided), which are subsequently coupled to some of the $E_{8(8)}$ invariances of the supergravity Lagrangian [11]. In that case there exists a large variety of gauge groups of rather high dimension. In $d = 4$ dimensions there are 28 vector gauge fields, but the $E_{7(7)}$ invariance is not reflected in the Lagrangian but only in the combined field equations and Bianchi identities by means of electric-magnetic duality. This duality rotates magnetic and electric charges, but the gauge couplings must be of the electric type. Then, in $d = 5$ dimensions, tensor and vector gauge fields are dual to one another in the absence of charges. The $E_{6(6)}$ invariance is only manifest when all the tensor fields have been converted to vector fields (transforming according to the 27-dimensional representation). In the presence of charges, however, the vector fields must either correspond to a nonabelian gauge group or they must be neutral. Charged would-be vector fields that do not correspond to the nonabelian gauge group, should be converted into antisymmetric tensor fields [12].

This implies that the field content of the $d = 5$ theory depends on the gauge group.

The Lagrangian of ungauged maximal supergravity contains the standard Einstein-Hilbert, Rarita-Schwinger and Dirac Lagrangians for the gravitons, the gravitini and the spinor fields. The kinetic terms of the gauge fields depend on the scalar fields and the kinetic term for the scalar fields takes the form of a nonlinear sigma model based on a symmetric coset space G/H. Here H is the maximal compact subgroup of G; a list of these groups is given in table 1. The Lagrangian (or the combined field equations and Bianchi identities) is invariant under the isometry group G which is referred to as the duality group. The standard treatment of gauged nonlinear sigma models exploits a formulation in which the group H is realized as a local invariance which acts on the spinor fields and the scalars; the corresponding connections are composite fields. The gauging is based on a gauge group $\mathrm{G}_g \subset \mathrm{G}$ whose connections are (some of the) elementary vector gauge fields of the supergravity theory. The matrix which encodes the embedding of the gauge group into the duality group is in fact linearly related to the T-tensor. The coupling constant associated with the gauge group G_g will be denoted by g. One can impose a gauge condition with respect to the local H invariance which amounts to fixing a coset representative for the coset space. In that case the G-symmetries will act nonlinearly on the fields and these nonlinearities make many calculations intractable or, at best, very cumbersome. Because it is much more convenient to work with symmetries that are realized linearly, the best strategy is therefore to postpone the gauge fixing till the end.

This paper aims at exploiting the group-theoretical constraints on the T-tensor, which are essential in order to have a consistent, supersymmetric gauging. It is well-known that the T-tensor must be restricted to a certain representation of the duality group. For instance, in four dimensions, this is the **912** representation of $\mathrm{E}_{7(7)}$, and in five dimensions it is the **351** representation of $\mathrm{E}_{6(6)}$. We derive these allowed representations for dimensions $d = 3, \ldots, 7$. Possible gaugings can then be explored by investigating which gauge groups lead to T-tensors that belong to the required representation. This proves to be sufficiently powerful to completely identify the possible gauge groups within a given subgroup of G, and to determine which gauge fields and generators of G are involved in each of the gaugings. Part of the analysis is done with help of the computer. To demonstrate the method and its potential, we analyze a number of gaugings in $d = 4, 5$ dimensions, including the known cases. Applications with hitherto unknown gauge groups are relegated to a forthcoming publication [13].

In four dimensions the Lagrangian is not unique in the absence of charges, because of electric/magnetic duality. Once the charges are switched on, the possibility of obtaining alternative Lagrangians is restricted, because electric charges cannot be converted to magnetic ones. Without introducing the gauging, there exist different Lagrangians (*i.e.*, not related by local field redefinitions) with different symmetry groups, whose field equations and Bianchi identities are equivalent and share the same invariance group. This feature makes the four-dimensional case more subtle to analyze and therefore considerable attention is given to this case. In particular, we show that the different Lagrangians of the ungauged theory are encoded in a matrix E belonging to $\mathrm{E}_{7(7)}\backslash\mathrm{Sp}(56;\mathbb{R})/\mathrm{GL}(28)$.

Some of the gaugings can be interpreted as originating from a Scherk-Schwarz truncation of a higher-dimensional theory [6]. In order to identify such gaugings we have included some material on these reductions and we exhibit examples in four and five dimensions. Both of them are single reductions, originating from a theory with one extra dimension. However, also multiple reductions are possible from theories with more than one extra dimensions, which lead to more complicated gauge groups, as we shall discuss in more detail in [13].

This paper is organized as follows. In section 2 we review the structure of the nonlinear sigma models that appear in maximal supergravity and the symmetries of the Lagrangians. In section 3 we focus on the definition of the T-tensor in four dimensions and discuss a large number of relevant features. In section 4, we derive the group theoretical constraint on the T-tensor (and equivalently on the embedding matrix of the gauge group) for dimensions $d = 3, \ldots, 7$. This constraint provides an efficient criterion for identifying consistent gaugings. In section 5 we review characteristic features of Scherk-Schwarz reductions, which correspond to some of the gaugings. Finally, in sections 6 and 7, we demonstrate how this framework naturally comprises the known gaugings in $d = 4$ and 5 dimensions. In addition, we exhibit a new gauging in $d = 5$ dimensions which can be interpreted as a Scherk-Schwarz reduction. An appendix is included with some group-theoretical results.

2 Coset-space geometry and duality

In this section we review the coset-space structure of the maximally supersymmetric supergravity theories. Because of the subtleties of supergravity in four spacetime dimensions special attention is devoted to this theory. In particular, we discuss its inequivalent Lagrangians, encoded in a matrix E. The existence of different Lagrangians makes the analysis of the various gaugings more complicated, as they are associated with different classes of gaugings.

The scalar fields in maximal supergravity parametrize a symmetric G/H coset space and the theory is realized with a local H symmetry with composite connection fields. The standard formalism starts from a matrix-valued field, $\mathcal{V}(x)$, that belongs to the group G, usually in the fundamental representation. Before introducing an (optional) gauging, this field transforms under rigid G transformations from the left and under local H transformations from the right. G-invariant one-forms are defined by

$$\mathcal{V}^{-1}\partial_\mu\mathcal{V} = \mathcal{Q}_\mu + \mathcal{P}_\mu\,, \tag{2.1}$$

where $\mathcal{Q}_\mu$ and $\mathcal{P}_\mu$ take their values in the Lie algebra associated with G; $\mathcal{Q}_\mu$ acts as a gauge field associated with the local H transformations. Eventually one may fix the gauge freedom associated with H, but until that point $\mathcal{V}$ will just be an unrestricted spacetime dependent element of the group G. After imposing the gauge condition on $\mathcal{V}(x)$ one obtains the coset representative $\mathcal{V}(\phi(x))$, where the fields $\phi(x)$ parametrize the coset space. The spinor fields of the supergravity Lagrangian transform under H, but are invariant under G. It is convenient to work with an H-covariant derivatives, which on $\mathcal{V}$ is equal to $D_\mu\mathcal{V} = \partial_\mu\mathcal{V} - \mathcal{V}\mathcal{Q}_\mu$, so that (2.1) can be written as,

$$\mathcal{V}^{-1}D_\mu\mathcal{V} = \mathcal{P}_\mu\,. \tag{2.2}$$

The quantities $\mathcal{Q}_\mu$ and $\mathcal{P}_\mu$ are subject to the Cartan-Maurer equations, which follow directly from (2.1),

$$\begin{aligned} F_{\mu\nu}(\mathcal{Q}) &= \partial_\mu\mathcal{Q}_\nu - \partial_\nu\mathcal{Q}_\mu + [\mathcal{Q}_\mu,\mathcal{Q}_\nu] = -[\mathcal{P}_\mu,\mathcal{P}_\nu]\,,\\ D_{[\mu}\mathcal{P}_{\nu]} &= \partial_{[\mu}\mathcal{P}_{\nu]} + [\mathcal{Q}_{[\mu},\mathcal{P}_{\nu]}] = 0\,. \end{aligned} \tag{2.3}$$

Here we made use of the fact that the generators of the subgroup H close and that the remaining generators associated with G/H form a representation of the group H. Furthermore, the commutators of any two of the latter generators are proportional to the generators of H. This last requirement is responsible for the zero on the right-hand side of the second Cartan-Maurer equation and ensures that we are dealing with a symmetric coset space. The Lagrangian of the corresponding nonlinear sigma model is invariant under both rigid G transformations and local H transformations and reads,

$$\mathcal{L} \propto \tfrac{1}{2}\mathrm{tr}\Big[D_\mu\mathcal{V}^{-1}\,D^\mu\mathcal{V}\Big] = -\tfrac{1}{2}\mathrm{tr}\Big[\mathcal{P}_\mu\,\mathcal{P}^\mu\Big]\,. \tag{2.4}$$

For maximal supergravity theories the symmetric cosets are known [15]. For the convenience of the reader we have listed them in table 1. Part of the isometry group can now be gauged by coupling the (elementary) vector gauge fields A_μ to a subset of the generators corresponding to a group $\mathrm{G}_g \subset \mathrm{G}$. The dimension of this gauge group is restricted by the number of available vector gauge fields. We have listed the field content for the bosonic fields assigned to representations of H in a second table 2. Because the gauge group G_g is embedded in the isometry group G, it must act on $\mathcal{V}$ so that the covariant derivative of $\mathcal{V}$ changes by the addition of the gauge fields A_μ which take their values in the Lie algebra corresponding to G_g,

$$D_\mu\mathcal{V}(x) = \partial_\mu\mathcal{V}(x) - \mathcal{V}(x)\,\mathcal{Q}_\mu(x) - g\,A_\mu(x)\,\mathcal{V}(x)\,. \tag{2.5}$$

d	G	H	dim [G] − dim [H]
11	1	1	$0-0=0$
10A	$\mathrm{SO}(1,1)/\mathbf{Z}_2$	1	$1-0=1$
10B	$\mathrm{SL}(2)$	$\mathrm{SO}(2)$	$3-1=2$
9	$\mathrm{GL}(2)$	$\mathrm{SO}(2)$	$4-1=3$
8	$\mathrm{E}_{3(+3)} \sim \mathrm{SL}(3)\times\mathrm{SL}(2)$	$\mathrm{U}(2)$	$11-4=7$
7	$\mathrm{E}_{4(+4)} \sim \mathrm{SL}(5)$	$\mathrm{USp}(4)$	$24-10=14$
6	$\mathrm{E}_{5(+5)} \sim \mathrm{SO}(5,5)$	$\mathrm{USp}(4)\times\mathrm{USp}(4)$	$45-20=25$
5	$\mathrm{E}_{6(+6)}$	$\mathrm{USp}(8)$	$78-36=42$
4	$\mathrm{E}_{7(+7)}$	$\mathrm{SU}(8)$	$133-63=70$
3	$\mathrm{E}_{8(+8)}$	$\mathrm{SO}(16)$	$248-120=128$

Table 1: Homogeneous scalar manifolds G/H for maximal supergravities in various dimensions. The type-IIB theory cannot be obtained from reduction of 11-dimensional supergravity and is included for completeness. The difference of the dimensions of G and H equals the number of scalar fields.

With this change, the expressions for $\mathcal{Q}_\mu$ and $\mathcal{P}_\mu$ are still given by (2.2), but the derivative is now covariantized and modified by the terms depending on the new gauge fields A_μ. The consistency of this procedure is obvious as (2.2) is fully covariant. Of course, the original rigid invariance under G transformations from the left is in general broken by the embedding of the new gauge group G_g into G.

The modifications caused by the new minimal couplings are minor and the effects can be concisely summarized by the Cartan-Maurer equations,

$$\begin{aligned} \mathcal{F}_{\mu\nu}(\mathcal{Q}) &= [\mathcal{P}_\mu, \mathcal{P}_\nu] - g\Big[\mathcal{V}^{-1}F_{\mu\nu}(A)\mathcal{V}\Big]_{\mathrm{H}}, \\ D_{[\mu}\mathcal{P}_{\nu]} &= -\tfrac{1}{2}g\Big[\mathcal{V}^{-1}F_{\mu\nu}(A)\mathcal{V}\Big]_{\mathrm{G/H}}. \end{aligned} \tag{2.6}$$

Note that $\mathcal{P}_\mu$ and $\mathcal{Q}_\mu$ are invariant under G_g (but transform under local H-transformations, as before).

In the following we concentrate on $d = 4$ spacetime dimensions [14, 1], where the Lagrangian is not uniquely defined and alternative Lagrangians, not related by local field redefinitions, can be obtained (in the absence of charges) via so-called electric-magnetic duality transformations (for a recent review, see [16]). These transformations constitute the group $\mathrm{Sp}(56;\mathbb{R})$. Lagrangians related via electric-magnetic duality do not share the same symmetry group and therefore they may allow different gaugings as the gauge group must be embedded into this group. Once the charges have been switched on, the possibilities for performing electric-magnetic duality are severely restricted, as electric charges cannot be converted to magnetic ones via local field redefinitions. This

d	$\mathrm{H_R}$	$p=0$	$p=1$	$p=2$	$p=3$	$p=4$
11	1	0	0	0	1	0
10A	1	1	1	1	1	0
10B	SO(2)	2	0	2	0	1^*
9	SO(2)	$2+1$	$2+1$	2	1	
8	U(2)	$5+1+\bar{1}$	$3+\bar{3}$	3	[1]	
7	USp(4)	14	10	5		
6	USp(4) $\times$ USp(4)	(5,5)	(4,4)	$(5,1)+(1,5)$		
5	USp(8)	42	27			
4	U(8)	$35+\overline{35}$	[28]			
3	SO(16)	128				

Table 2: Bosonic field content for maximal supergravities described by p-rank antisymmetric gauge fields; $p=0$ corresponds to a scalar field and the graviton fields has been suppressed. The $p=4$ gauge field in $d=10\mathrm{B}$ has a self-dual field strength. The representations [1] and [28] (in $d=8,4$, respectively) are extended to U(1) and SU(8) representations through duality transformations on the field strengths. These transformations can not be represented on the vector potentials. In $d=3$ dimensions, the graviton does not describe propagating degrees of freedom. For $p>0$ the fields can be assigned to representations of a bigger group than $\mathrm{H_R}$.

complication is specific to 4 dimensions; for $d \neq 4$ the situation is simpler and the results of this section can be taken over without much difficulty.

For $d=4$, $\mathcal{V}(x)$ is a 56×56 matrix, sometimes called the 56-bein, which decomposes as follows,

$$\mathcal{V}(x) = \begin{pmatrix} u^{ij}{}_{IJ}(x) & -v_{klIJ}(x) \\ & \\ -v^{ijKL}(x) & u_{kl}{}^{KL}(x) \end{pmatrix} . \tag{2.7}$$

The indices $I,J,\ldots$ and $i,j,\ldots$ take the values $1,\ldots,8$, so that there are 28 anti-symmetrized index pairs representing the matrix indices of $\mathcal{V}$; the row indices are $([IJ],[KL])$, and the column indices are $([ij],[kl])$, so as to remain consistent with the conventions of [1]. The above matrix is pseudoreal and belongs to $\mathrm{E}_{7(7)} \subset \mathrm{Sp}(56;\mathbb{R})$ in the fundamental representation.[1] We use the convention where $u^{ij}{}_{IJ} = (u_{ij}{}^{IJ})^*$ and $v_{ijIJ} = (v^{ijIJ})^*$. The indices $i,j,\ldots$ refer to SU(8) and capital indices $I,J,\ldots$ are subject to $\mathrm{E}_{7(7)}$ transformations. Using the above definition, one may evaluate the

[1]The pseudoreal representation stresses the maximal compact SU(8) subgroup. It would perhaps be appropriate to denote the pseudoreal representation by USp(28, 28), but for reasons of uniformity we will always refer to $\mathrm{Sp}(56;\mathbb{R})$.

quantities $\mathcal{Q}_\mu$ and $\mathcal{P}_\mu$,

$$\mathcal{V}^{-1}\partial_\mu\mathcal{V} = \begin{pmatrix} \mathcal{Q}_{\mu\,ij}{}^{mn} & \mathcal{P}_{\mu\,ijpq} \\ \\ \mathcal{P}_\mu^{klmn} & \mathcal{Q}_\mu{}^{kl}{}_{pq} \end{pmatrix} , \tag{2.8}$$

which leads to the expressions,

$$\begin{aligned} \mathcal{Q}_{\mu\,ij}{}^{kl} &= u_{ij}{}^{IJ}\,\partial_\mu u^{kl}{}_{IJ} - v_{ijIJ}\,\partial_\mu v^{klIJ} , \\ \mathcal{P}_\mu^{ijkl} &= v^{ijIJ}\,\partial_\mu u^{kl}{}_{IJ} - u^{ij}{}_{IJ}\,\partial_\mu v^{klIJ} . \end{aligned} \tag{2.9}$$

Compatibility with the Lie algebra of $\mathrm{E}_{7(7)}$ implies that $\mathcal{P}_\mu^{ijkl}$ is a selfdual SU(8) tensor,

$$\mathcal{P}_\mu^{ijkl} = \tfrac{1}{24}\,\varepsilon^{ijklmnpq}\,\mathcal{P}_{\mu\,mnpq} , \tag{2.10}$$

and $\mathcal{Q}_\mu$ transforms as a connection associated with SU(8). Hence, $\mathcal{Q}_{\mu\,ij}{}^{kl}$ satisfies the decomposition,

$$\mathcal{Q}_{\mu\,ij}{}^{kl} = \delta_{[i}^{[k}\,\mathcal{Q}_{\mu\,j]}{}^{l]} , \tag{2.11}$$

with

$$\mathcal{Q}_{\mu\,i}{}^{j} = \tfrac{2}{3}\Big[u_{ik}{}^{IJ}\,\partial_\mu u^{jk}{}_{IJ} - v_{ikIJ}\,\partial_\mu v^{jkIJ}\Big] , \tag{2.12}$$

and $\mathcal{Q}_{\mu}{}^{i}{}_{j} = -\mathcal{Q}_{\mu j}{}^{i}$ and $\mathcal{Q}_{\mu i}{}^{i} = 0$.

While the index pairs $[IJ]$ refer to the row indices of $\mathcal{V}$ and are subject to $\mathrm{E}_{7(7)}$, the 28 gauge fields A_μ^{AB} are labelled by index pairs $[AB]$, where $A, B = 1,\ldots,8$. As it turns out [17], the ungauged Lagrangians can be encoded into a matrix E belonging to $\mathrm{E}_{7(7)}\backslash\mathrm{Sp}(56;\mathbb{R})/\mathrm{GL}(28)$, which defines the embedding of the 28 vector fields into the 56-bein and thus connects the two types of index pairs $[IJ]$ and $[AB]$,[2]

$$\mathsf{E} = \begin{pmatrix} \mathsf{U}_{IJ}{}^{AB} & \mathsf{V}_{IJCD} \\ \\ \mathsf{V}^{KLAB} & \mathsf{U}^{KL}{}_{CD} \end{pmatrix} . \tag{2.13}$$

Two Lagrangians related by electric-magnetic duality correspond to two matrices E related by multiplication from the left by an element of $\mathrm{Sp}(56;\mathbb{R})$. These matrices are not unique, because an $\mathrm{E}_{7(7)}$ transformation can always be absorbed into the 56-bein and a $\mathrm{GL}(28;\mathbb{R})$ transformation can be absorbed into the gauge fields. It is convenient to include E into the 56-bein according to,

$$\hat{\mathcal{V}}(x) = \mathsf{E}^{-1}\,\mathcal{V}(x) , \tag{2.14}$$

where we have to remember that $\hat{\mathcal{V}}$ is now no longer a group element of $\mathrm{E}_{7(7)}$! This definition leads to corresponding submatrices $u^{ij}{}_{AB}$ and v^{ijAB}.

[2]Similar additional parameters in four-dimensional Lagrangians have been exploited also in $N = 2, 4$ supergravity [18, 19].

Although the $\mathrm{E}_{7(7)}$ tensors $\mathcal{Q}_\mu$ and $\mathcal{P}_\mu$ are not affected by the matrix E and have identical expressions in terms of $\mathcal{V}$ and $\hat{\mathcal{V}}$, the remaining interactions depend on E, and so do the transformation rules. However, by making use of (2.14) we can make the dependence on E implicit, provided we also introduce an SU(8) (selfdual) covariant field strength, defined by

$$F^{+AB}_{\mu\nu} = (u^{ij}{}_{AB} + v^{ijAB})\,\mathsf{F}^{+}_{\mu\nu ij} - (u_{ij}{}^{AB} + v_{ijAB})\,\mathcal{O}^{+ij}_{\mu\nu}\,, \tag{2.15}$$

where $F^{AB}_{\mu\nu} = 2\partial_{[\mu}A^{AB}_{\nu]}$, and $\mathcal{O}^{+ij}_{\mu\nu}$ is a selfdual Lorentz tensor that comprises terms quadratic in the fermion fields. The anti-selfdual tensors are obtained by complex conjugation. The terms in the Lagrangian that depend on the field strengths, take the form

$$\begin{aligned}\mathcal{L} \;=\; & -\tfrac{1}{8}e\,\mathcal{N}_{AB,CD}\,F^{+AB}_{\mu\nu}\,F^{+CD\,\mu\nu} - \tfrac{1}{2}e\,F^{+AB}_{\mu\nu}\,[(u+v)^{-1}]^{AB}{}_{ij}\,\mathcal{O}^{+\mu\nu\,ij} \\ & + \text{h.c.}\,,\end{aligned} \tag{2.16}$$

where the complex 28×28 symmetric matrix $\mathcal{N}$ is defined by $(u^{ij}{}_{AB}+v^{ijAB})\,\mathcal{N}_{AB,CD} = u^{ij}{}_{CD} - v^{ijCD}$. Obviously, (2.16) depends only implicitly on E.

The choice for E has a bearing on the manifest subgroup of $\mathrm{E}_{7(7)}$ under which the Lagrangian is invariant. We shall call this group the *electric* duality group G_e and define it as the largest subgroup of $\mathrm{E}_{7(7)}$ which acts on all the fields, including the 28 gauge fields A^{AB}_μ, that leaves the action invariant. Defining $G_{\mu\nu\,AB} \propto \varepsilon_{\mu\nu\rho\sigma}\,\partial\mathcal{L}/\partial F^{AB}_{\rho\sigma}$ the group G_e acts as follows,

$$\begin{aligned}\delta F^{AB}_{\mu\nu} &= \tilde\Lambda^{AB}{}_{CD}\,F^{CD}_{\mu\nu}\,, \\ \delta G_{\mu\nu\,AB} &= -\tilde\Lambda^{CD}{}_{AB}\,G_{\mu\nu\,CD} + \tilde\Sigma_{ABCD}\,F^{CD}_{\mu\nu}\,,\end{aligned} \tag{2.17}$$

with $\tilde\Lambda^{AB}{}_{CD}$ and $\tilde\Sigma_{ABCD} = \tilde\Sigma_{CDAB}$ real. Obviously, the $\tilde\Lambda$ characterize the transformations of the vector potentials A^{AB}_μ; the part of the generators that resides in $\tilde\Sigma$ is realized in the transformations of $\mathcal{V}$. The variations (2.17) generate the subgroup $\mathrm{G}_\mathrm{e}\subset \mathrm{E}_{7(7)}$. However, we use a formulation based on the complex combinations $(iG\pm F)_{\mu\nu}$ which is connected to the $\mathrm{E}_{7(7)}$ basis for $\mathcal{V}$ via the matrix E. This is the basis that is relevant for $u^{ij}{}_{AB}$ and v_{ijAB}, which transform according to

$$\begin{aligned}\delta u_{ij}{}^{AB} &= -u_{ij}{}^{CD}\,\Lambda_{CD}{}^{AB} - v_{ijCD}\,\Sigma^{CDAB}\,, \\ \delta v^{ijAB} &= -v^{ijCD}\,\Lambda_{CD}{}^{AB} - u^{ij}{}_{CD}\,\Sigma^{CDAB}\,,\end{aligned} \tag{2.18}$$

where

$$\begin{aligned}\Lambda_{AB}{}^{CD} = (\Lambda^{AB}{}_{CD})^* &= \tfrac{1}{2}(\tilde\Lambda^{AB}{}_{CD} - \tilde\Lambda^{CD}{}_{AB} + i\tilde\Sigma_{ABCD})\,, \\ \Sigma_{ABCD} = (\Sigma^{ABCD})^* &= -\tfrac{1}{2}(\tilde\Lambda^{AB}{}_{CD} + \tilde\Lambda^{CD}{}_{AB} + i\tilde\Sigma_{ABCD})\,.\end{aligned} \tag{2.19}$$

Clearly these parameters are subject to the constraint

$$\mathrm{Im}\,\Big(\Sigma_{ABCD} + \Lambda_{AB}{}^{CD}\Big) = 0\,. \tag{2.20}$$

Under these transformations we find the following results,

$$\begin{aligned}
\delta(u_{ij}{}^{AB} + v_{ijAB}) &= \tilde\Lambda^{AB}{}_{CD}\,(u_{ij}{}^{CD} + v_{ijCD})\ , \\
\delta(u^{ij}{}_{AB} - v^{ijAB}) &= -\tilde\Lambda^{CD}{}_{AB}\,(u^{ij}{}_{CD} - v^{ijCD}) + \tilde\Sigma_{ABCD}\,(u^{ij}{}_{CD} + v^{ijCD})\,, \\
\delta\mathcal{N}_{AB,CD} &= -\tilde\Lambda^{EF}{}_{AB}\,\mathcal{N}_{EF,CD} - \mathcal{N}_{AB,EF}\,\tilde\Lambda^{EF}{}_{CD} + i\tilde\Sigma_{ABCD}\,.
\end{aligned} \tag{2.21}$$

It is now easy to verify that the Lagrangian (2.16) is not invariant under these transformations but changes instead into a total derivative,

$$\delta\mathcal{L} = \tfrac{1}{16}\varepsilon^{\mu\nu\rho\sigma}\,\tilde\Sigma_{ABCD}\,F_{\mu\nu}^{AB}\,F_{\rho\sigma}^{CD}\,. \tag{2.22}$$

Transformations with $\tilde\Sigma \neq 0$ induce a shift in the generalized theta angle and are therefore called Peccei-Quinn transformations. When the Peccei-Quinn transformations are part of a nonabelian gauge group associated with the gauge fields A_μ^{AB}, so that the corresponding $\tilde\Sigma$ depends on the spacetime coordinates, then (2.22) is no longer a total derivative. In that case one must include a Chern-Simons-like term

$$\mathcal{L} \propto g\,\varepsilon^{\mu\nu\rho\sigma}\,\tilde\Sigma_{ABCD;EF}\,A_\mu^{EF}\,A_\nu^{AB}\,(\partial_\rho A_\sigma^{CD} - \tfrac{1}{8}g\,f_{GH,IJ}{}^{CD}A_\rho^{GH}A_\sigma^{IJ})\,, \tag{2.23}$$

where the constants $g\,\tilde\Sigma_{ABCD;EF}$ are the coefficients that one obtains when expanding $\tilde\Sigma$ in terms of the gauge group parameters $\Xi^{EF}(x)$, and the $f_{AB,CD}{}^{EF}$ are the structure constants of the gauge group. The addition of the term (2.23) not only restores the gauge invariance of the action, but also the supersymmetry, as was shown in [20]. The fact that the constants $\tilde\Sigma_{ABCD;EF}$ emerge as variations of the tensor $\mathcal{N}$ under the action of a nonabelian group, implies that they are subject to certain constraints. These constraints allow only nontrivial solutions when the group is non-semisimple [21].

Without the condition (2.20), the variations (2.18) define an infinitesimal $\mathrm{Sp}(56;\mathbb{R})$ transformation in the pseudoreal basis. In terms of (2.17) the extra contributions are related to an extra variation of $\delta F_{\mu\nu} \propto G_{\mu\nu}$ proportional to a second real parameter $\tilde\Sigma'$. Its effect is to shift the contribution of $\tilde\Sigma$ in the two equations in (2.19) by equal but opposite amounts, such that (2.20) is no longer satisfied. The maximal compact subgroup of $\mathrm{Sp}(56;\mathbb{R})$ resides in the $\Lambda_{AB}{}^{CD}$ and is equal to U(28).

In the following we distinguish some classes of Lagrangians with inequivalent symmetry groups $\mathrm{G_e}$ and corresponding matrices E. Different matrices E are related by electric-magnetic duality transformations, which can be worked out explicitly for any given case. We concentrate on the semisimple subgroups of $\mathrm{G_e}$. Obviously the group $\mathrm{G_e}$ must have a *real* 28-dimensional representation. We will discuss four inequivalent cases, based on the semisimple groups $\mathrm{SL}(8,\mathbb{R})$, $\mathrm{E}_{6(6)}\times\mathrm{SO}(1,1)$, $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(1,1)\times\mathrm{SL}(6,\mathbb{R})$, and $\mathrm{SU}^\star(8)$.

The $\mathrm{SL}(8,\mathbb{R})$**-basis:** $\mathsf{E} = \mathsf{E}_{\mathrm{SL}(8,\mathbb{R})} = \mathrm{Id}$ The Lagrangian is invariant under $\mathrm{SL}(8,\mathbb{R})$, and the gauge fields transform in the **28** representation. In terms of (2.17), the generators are associated with $\tilde{\Lambda}$, whereas (2.20) is satisfied in view of the fact that both matrices Λ and Σ are real. The **56** representation of $\mathrm{E}_{7(7)}$ decomposes into the $\mathbf{28}+\overline{\mathbf{28}}$ representation of $\mathrm{SL}(8,\mathbb{R})$. The corresponding Lagrangian is the one written down in [1]. To understand the relation with the alternative, but inequivalent, Lagrangians discussed below, we decompose the **28** and $\overline{\mathbf{28}}$ representation pertaining to the vector fields and their magnetic duals, according to the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(1,1) \times \mathrm{SL}(6,\mathbb{R})$ subgroup,

$$\begin{aligned} \mathbf{28} &\rightarrow (\mathbf{2},\mathbf{6})_{-1} + (\mathbf{1},\mathbf{15})_{+1} + (\mathbf{1},\mathbf{1})_{-3}\,, \\ \overline{\mathbf{28}} &\rightarrow (\mathbf{2},\overline{\mathbf{6}})_{+1} + (\mathbf{1},\overline{\mathbf{15}})_{-1} + (\mathbf{1},\mathbf{1})_{+3}\,. \end{aligned} \tag{2.24}$$

There are no other symmetries of the Lagrangian beyond $\mathrm{SL}(8,\mathbb{R})$. Of course, the combined field equations and Bianchi identities have $\mathrm{E}_{7(7)}$ as a symmetry group, but this is true in general; it is only the symmetry group of the Lagrangian that can differ, depending on the choice for E.

The $\mathrm{E}_{6(6)}$**-basis:** $\mathsf{E} = \mathsf{E}_{\mathrm{E}_{6(6)}}$ For a different choice of E discussed below, there exists a larger group $\mathrm{G_e}$ which contains the semisimple group $\mathrm{E}_{6(6)} \times \mathrm{SO}(1,1) \subset \mathrm{E}_{7(7)}$ as a subgroup. The latter corresponds to the generators $\tilde{\Lambda}$ in(2.17) and acts block diagonally (the blocks Λ, Σ are real). In addition, there are 27 nilpotent generators which reside in both $\tilde{\Lambda}$ and $\tilde{\Sigma}$ and extend the group to a non-semisimple one. They transform in the $\overline{\mathbf{27}}_{+2}$ representation of $\mathrm{E}_{6(6)} \times \mathrm{SO}(1,1)$ and give rise to the imaginary parts of Λ and Σ, although their combined contribution to (2.20) vanishes. This $\mathrm{E}_{6(6)}$-basis is based on a matrix E such that the Lagrangian coincides with the Lagrangian that one obtains upon reduction to four dimensions of five-dimensional ungauged maximally supersymmetric supergravity.

To understand the difference between this basis and the previous one, let us consider the action of the common subgroup $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(1,1)\times\mathrm{SL}(6,\mathbb{R}) = \big(\mathrm{E}_{6(6)} \times \mathrm{SO}(1,1)\big)\cap \mathrm{SL}(8,\mathbb{R})$. With respect to $\mathrm{E}_{6(6)} \times \mathrm{SO}(1,1)$ the **56** of $\mathrm{E}_{7(7)}$ decomposes as:

$$\mathbf{56} = \overline{\mathbf{27}}_{-1} + \mathbf{1}_{-3} + \mathbf{27}_{+1} + \mathbf{1}_{+3}\,, \tag{2.25}$$

where the negative grading identifies the vector gauge fields and the positive grading their magnetic duals. According to the decomposition (2.17) the generators of $\mathrm{G_e}$ have the form [7],

$$\mathrm{E}_{6(6)} \;:\; \begin{pmatrix} K_{27} & 0 & \varnothing_{27} & 0 \\ 0 & 0 & 0 & 0 \\ \varnothing_{27} & 0 & -K_{27}^{\mathrm{T}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\,, \qquad \mathrm{SO}(1,1) : \begin{pmatrix} -\mathbb{1}_{27} & 0 & \varnothing_{27} & 0 \\ 0 & -3 & 0 & 0 \\ \varnothing_{27} & 0 & \mathbb{1}_{27} & 0 \\ 0 & 0 & 0 & +3 \end{pmatrix}\,,$$

$$\overline{\mathbf{27}}_{+2} \quad : \quad \begin{pmatrix} \varnothing_{27} & \vec{t} & \varnothing_{27} & 0 \\ 0 & 0 & 0 & 0 \\ L_{27} & 0 & \varnothing_{27} & 0 \\ 0 & 0 & -\vec{t}^{\,\mathrm{T}} & 0 \end{pmatrix} , \tag{2.26}$$

where $\vec{t}$ denotes a 27-dimensional vector of parameters, the subscript 27 denotes 27×27 matrices, and K_{27} denotes the generators of $\mathrm{E}_{6(6)}$ in the $\overline{\mathbf{27}}$ representation; the symmetric matrix L_{27} is expressed as $L_{\Lambda\Sigma} = d_{\Lambda\Sigma\Gamma}\, t^{\Gamma}$, where $d_{\Lambda\Sigma\Gamma}$ defines the cubic invariant in the $\overline{\mathbf{27}}$ representation of $\mathrm{E}_{6(6)}$. According to (2.25), the 28 vector potentials A_μ^{AB} now transform in the $\overline{\mathbf{27}}_{-1} + \mathbf{1}_{-3}$ representation of $\mathrm{E}_{6(6)} \times \mathrm{SO}(1,1)$, which decomposes with respect to the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(1,1) \times \mathrm{SL}(6,\mathbb{R})$ subgroup according to

$$\overline{\mathbf{27}}_{-1} + \mathbf{1}_{-3} \quad \rightarrow \quad (\mathbf{2},\mathbf{6})_{-1} + (\mathbf{1},\overline{\mathbf{15}})_{-1} + (\mathbf{1},\mathbf{1})_{-3}\,. \tag{2.27}$$

Comparing this result with the decomposition (2.24) shows that the matrix E corresponds therefore to the duality transformation that interchanges the $(\mathbf{1},\mathbf{15})_{+1}$ gauge fields of the $\mathrm{SL}(8,\mathbb{R})$-basis with the corresponding dual gauge fields belonging to the $(\mathbf{1},\overline{\mathbf{15}})_{-1}$ representation. This duality transformation is not a symmetry of the Lagrangian and the matrix E is thus not of the form (2.17). In the pseudoreal basis E is block diagonal and $\mathsf{U}_{IJ}{}^{AB}$ acts as $i\mathbb{1}_{15}$ on the subspace spanned by $(\mathbf{1},\mathbf{15})_{+1}$ (the sign is irrelevant) and as the identity on $(\mathbf{2},\mathbf{6})_{-1} + (\mathbf{1},\mathbf{1})_{-3}$. This does not constitute an element of $\mathrm{E}_{7(7)}$. If that were the case the matrix U should be an element of $\mathrm{SU}(8)$ acting in the $\mathbf{28}$ representation. However, there is only one nontrivial matrix U that is diagonal with eigenvalues 1 and/or $+i$, namely, the matrix diag $(\mathbb{1}_{12}, i\mathbb{1}_{16})$. Hence we conclude that E is indeed a nontrivial element of $\mathrm{E}_{7(7)}\backslash\mathrm{Sp}(56;\mathbb{R})/\mathrm{GL}(28)$. Let us denote this transformation by $\mathsf{E}_{\mathrm{E}_{6(6)}}$.

The $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(1,1) \times \mathrm{SL}(6,\mathbb{R})$**-basis:** $\mathsf{E} = \mathsf{E}_{\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(1,1)\times\mathrm{SL}(6,\mathbb{R})}$ This basis is related to the two previous ones upon performing an additional electric-magnetic duality transformation, this time also interchanging the $(\mathbf{2},\mathbf{6})_{-1}$ gauge fields of the $\mathrm{SL}(8,\mathbb{R})$-basis with their dual gauge fields in the $(\mathbf{2},\overline{\mathbf{6}})_{+1}$ representation, so that the gauge fields decompose according to

$$(\mathbf{2},\overline{\mathbf{6}})_{+1} + (\mathbf{1},\overline{\mathbf{15}})_{-1} + (\mathbf{1},\mathbf{1})_{-3}\,. \tag{2.28}$$

The semisimple invariance group $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(1,1) \times \mathrm{SL}(6,\mathbb{R})$ is extended by 12 nilpotent generators, which belong to the $(\mathbf{2},\mathbf{6})_{+2}$ representation, so that $\mathrm{G_e}$ is again a non-semisimple group. To derive the above results is not difficult. One simply decomposes the possible generators of $\mathrm{E}_{7(7)}$ in this duality rotated basis in terms of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SO}(1,1) \times \mathrm{SL}(6,\mathbb{R})$ and verifies which generators satisfy (2.17) or (2.20). The new matrix E is equal to $\mathsf{E}'\,\mathsf{E}_{\mathrm{E}_{6(6)}}$, where E' is again a block-diagonal matrix in which the submatrix U is diagonal: diag $(\mathbb{1}_{16}, i\mathbb{1}_{12})$, which, according to the argument presented above, is not an element of SU(8), so that the new matrix E belongs to another equivalence class.

The $\mathrm{SU}^\star(8)$**-basis:** $\mathsf{E} = \mathsf{E}_{\mathrm{SU}^\star(8)}$ Another basis considered in the literature [8], is the one in which $\mathrm{G_e} = \mathrm{SU}^\star(8)$, the group of 8×8 matrices that are real up to a symplectic matrix. This group is generated inside $\mathrm{E}_{7(7)}$ by the generators of the maximal compact subgroup USp(8) of $\mathrm{E}_{6(6)}$ and by the non-compact part of the $\overline{\mathbf{27}}_{+2}$ generators defined above. The vector potentials transform in the **28** pseudoreal representation of $\mathrm{SU}^\star(8)$. The matrix $\mathsf{E}_{\mathrm{SU}^\star(8)}$ which realizes the transformation from the $\mathrm{SL}(8,\mathbb{R})$-basis to the $\mathrm{SU}^\star(8)$-basis can be expressed in the real representation by means of the following transformation:

$$\begin{aligned}
\mathsf{E}_{\mathrm{SU}^\star(8)} &= \mathsf{E}''\,\mathsf{E}_{\mathrm{E}_{6(6)}} \in \mathrm{Sp}(56,\mathbb{R})\,,\\
\mathsf{E}'' &= \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbb{1}_{27} & 0 & -\mathbb{1}_{27} & 0\\ 0 & 1 & 0 & 1\\ \mathbb{1}_{27} & 0 & \mathbb{1}_{27} & 0\\ 0 & -1 & 0 & 1\end{pmatrix}\,,
\end{aligned} \tag{2.29}$$

where the matrix E'' acts in the $\mathrm{E}_{6(6)}$-basis of (2.26).

3 The T-tensor

The gauging of supergravity is effected by switching on the gauge coupling constant, after assigning the various fields to representations of the gauge group embedded in G. Again we mainly focus on the four-dimensional theory, where $\mathrm{G} = \mathrm{E}_{7(7)}$, but we will occasionally comment on other space-time dimensions. In $d = 4$ dimensions only the gauge fields themselves and the spinless fields transform under the gauge group. In other dimensions, the ungauged supergravity theory has tensor gauge fields transforming in representations of the group G. Coupling the vector gauge fields to tensor gauge fields usually leads to a loss of tensor gauge invariance and may require changes of the field representations. We already alluded to this previously. However, for the analysis of this section, the tensor gauge fields play no role and we concentrate on the vector gauge fields and the scalar fields.

When switching on the charges, the abelian field strengths are changed to non-abelian ones and derivatives of the scalars are covariantized according to (*c.f.* (2.5))

$$\partial_\mu\mathcal{V} \to \partial_\mu\mathcal{V} - gA_\mu^{AB}\,t_{AB}\mathcal{V}\,, \tag{3.1}$$

where the gauge group generators t_{AB} span a subalgebra of the Lie algebra of $\mathrm{G_e} \subset \mathrm{E}_{7(7)}$ in the **56** representation, whose dimension is at most equal to the number of vector fields. Introducing the gauging causes a loss of supersymmetry, because the new terms in the Lagrangian yield new variations. The leading variations are induced by the modification (3.1) of the Cartan-Maurer equations. This modification was already noted in (2.6) and takes the form

$$\begin{aligned} F_{\mu\nu}(\mathcal{Q})_i{}^j &= -\tfrac{4}{3}\,\mathcal{P}_{[\mu}{}^{jklm}\,\mathcal{P}_{\nu]iklm} - g\,F_{\mu\nu}^{AB}\,\mathcal{Q}_{AB\,i}{}^j\,, \\ D_{[\mu}\mathcal{P}_{\nu]}^{ijkl} &= -\tfrac{1}{2}g\,F_{\mu\nu}^{AB}\,\mathcal{P}_{AB}^{ijkl}\,, \end{aligned} \tag{3.2}$$

where

$$\mathcal{V}^{-1}t_{AB}\mathcal{V} = \begin{pmatrix} \mathcal{Q}_{AB\,ij}{}^{mn} & \mathcal{P}_{AB\,ijpq} \\ & \\ \mathcal{P}_{AB}^{klmn} & \mathcal{Q}_{AB}{}^{kl}{}_{pq} \end{pmatrix}\,. \tag{3.3}$$

These modifications are the result of the implicit dependence of $\mathcal{Q}_\mu$ and $\mathcal{P}_\mu$ on the vector potentials A_μ^{AB}. This dependence can be expressed as follows,

$$\mathcal{Q}_\mu = \mathcal{Q}_\mu^{(0)} - gA_\mu^{AB}\,\mathcal{Q}_{AB}\,, \qquad \mathcal{P}_\mu = \mathcal{P}_\mu^{(0)} - gA_\mu^{AB}\,\mathcal{P}_{AB}\,. \tag{3.4}$$

The fact that the matrices t_{AB} generate a subalgebra of the algebra associated with $\mathrm{E}_{7(7)}$, implies that the quantities $\mathcal{Q}_{AB}$ and $\mathcal{P}_{AB}$ satisfy the constraints,

$$\begin{aligned} \mathcal{P}_{AB}^{ijkl} &= \tfrac{1}{24}\,\varepsilon^{ijklmnpq}\,\mathcal{P}_{AB\,mnpq}\,, \\ \mathcal{Q}_{AB\,ij}{}^{kl} &= \delta_{[i}^{[k}\,\mathcal{Q}_{AB\,j]}{}^{l]}\,, \end{aligned} \tag{3.5}$$

while $\mathcal{Q}_{ABi}{}^j$ is antihermitean and traceless. It is straightforward to write down the explicit expressions for $\mathcal{Q}_{AB}$ and $\mathcal{P}_{AB}$,

$$\begin{aligned} \mathcal{Q}_{AB\,i}{}^j &= \tfrac{2}{3}\Big[u_{ik}{}^{IJ}\,(\Delta_{AB}u^{jk}{}_{IJ}) - v_{ikIJ}\,(\Delta_{AB}v^{jkIJ})\Big]\,, \\ \mathcal{P}_{AB}^{ijkl} &= v^{ijIJ}\,(\Delta_{AB}u^{kl}{}_{IJ}) - u^{ij}{}_{IJ}(\Delta_{AB}v^{klIJ})\,. \end{aligned} \tag{3.6}$$

where $\Delta_{AB}u$ and $\Delta_{AB}v$ indicate the change of submatrices in $\mathcal{V}$ induced by left multiplication with the generator t_{AB}.

The presence of the order-g modifications in the Cartan-Maurer equations leads to new supersymmetry variations of the gravitino kinetic terms and the Noether term. These variations are proportional to the field strengths $F_{\mu\nu}^{AB}$, which must be re-expressed in terms of the SU(8) field strengths (2.15) in order to cancel against new supersymmetry variations and terms in the Lagrangian. Hence the order-g variations are proportional to the so-called T-tensor, which decomposes into two reducible representations of SU(8) that appear in the variations linear in the gravitino fields and linear in the spin-$\frac{1}{2}$ fields, respectively,

$$\begin{aligned} T_i^{jkl} &= \tfrac{3}{4}\mathcal{Q}_{AB\,i}{}^j\,(u^{kl}{}_{AB} + v^{klAB}) \\ &= \tfrac{1}{2}\Big[u_{im}{}^{IJ}\,(\Delta_{AB}u^{jm}{}_{IJ}) - v_{imIJ}\,(\Delta_{AB}v^{jmIJ})\Big](u^{kl}{}_{AB} + v^{klAB})\,, \end{aligned} \tag{3.7}$$

$$\begin{aligned} T_{ijkl}^{mn} &= \tfrac{1}{2}\mathcal{P}_{AB\,ijkl}\,(u^{mn}{}_{AB} + v^{mnAB}) \\ &= \tfrac{1}{2}\Big[v_{ijIJ}\,(\Delta_{AB}u_{kl}{}^{IJ}) - u_{ij}{}^{IJ}(\Delta_{AB}v_{klIJ})\Big]\,(u^{mn}{}_{AB} + v^{mnAB})\,. \end{aligned} \tag{3.8}$$

The T-tensor is thus a cubic product of the 56-bein $\mathcal{V}$ and depends in a nontrivial way on the embedding of the gauge group into $\mathrm{E}_{7(7)}$. It must satisfy a number of important properties which are generic and apply to arbitrary spacetime dimensions (of course, after switching to the corresponding G/H coset space). These properties are discussed below.

First we observe that any variation of $\mathcal{V}$ can be parametrized by

$$\mathcal{V} \to \mathcal{V} \begin{pmatrix} 0 & \overline{\Sigma} \\ \Sigma & 0 \end{pmatrix} , \tag{3.9}$$

up to a (local) SU(8) transformation. Under this variation one can easily show that the SU(8) tensors $\mathcal{Q}_{AB}$ and $\mathcal{P}_{AB}$ combine into the **133** representation of $\mathrm{E}_{7(7)}$. Likewise we can derive,

$$\begin{aligned} \delta T_i^{jkl} &= \Sigma^{jmnp}\, T^{kl}_{imnp} - \tfrac{1}{24}\varepsilon^{jmnpqrst}\,\Sigma_{imnp}\, T^{kl}_{qrst} + \Sigma^{klmn}\, T^j_{imn} \,, \\ \delta T^{mn}_{ijkl} &= \tfrac{4}{3}\Sigma_{p[ijk}\, T^{pmn}_{l]} - \tfrac{1}{24}\varepsilon_{ijklpqrs}\,\Sigma^{mntu}\, T^{pqrs}_{tu} \,. \end{aligned} \tag{3.10}$$

This shows that the SU(8) covariant T-tensors constitute a representation of $\mathrm{E}_{7(7)}$, corresponding to the right multiplication of $\mathcal{V}$. This property will play an important role below.

The T-tensors are subject to quadratic equations, which are crucial in establishing the supersymmetry at order g^2. Following [1] we first write $\hat{\mathcal{V}}\,\hat{\mathcal{V}}^{-1} = \mathbb{1}$ in terms of the submatrices u and v, and derive

$$(u^{ij}{}_{CD} + v^{ijCD})u_{ij}{}^{AB} = (u_{ij}{}^{CD} + v_{ijCD})v^{ijAB} + \delta^{AB}_{CD} \,. \tag{3.11}$$

Multiplication with $\mathcal{Q}_{CD}$ and $\mathcal{P}_{CD}$ and taking suitable linear combinations yields,

$$\begin{aligned} T^k_{lij}\,(u^{ij}{}_{AB} + v^{ijAB}) &= -T^{kij}_l\,(u_{ij}{}^{AB} + v_{ijAB}) \,, \\ T^{mn}_{ijkl}\,(u_{mn}{}^{AB} + v_{mnAB}) &= \tfrac{1}{24}\varepsilon_{ijklpqrs}\, T^{pqrs}_{mn}\,(u^{mn}{}_{AB} + v^{mnAB}) \,. \end{aligned} \tag{3.12}$$

Contracting once more with $\mathcal{Q}_{AB}$ and $\mathcal{P}_{AB}$ yields a number of identities quadratic in the T-tensors,

$$\begin{aligned} T^k_{lij}\, T^{mij}_n - T^{kij}_l\, T^m_{nij} &= 0 \,, \\ T^k_{lij}\, T^{ij}_{mnpq} + \tfrac{1}{24}\varepsilon_{mnpqrstu}\, T^{kij}_l\, T^{rstu}_{ij} &= 0 \,, \\ T^{vw}_{irst}\, T^{jrst}_{vw} - \tfrac{1}{8}\delta^j_i\, T^{vw}_{rstu}\, T^{rstu}_{vw} &= 0 \,, \\ T^{vw}_{ijkr}\, T^{mnpr}_{vw} - \tfrac{9}{4}\delta^{[m}_{[i}\, T^{vw}_{jk]rs}\, T^{np]rs}_{vw} + \tfrac{1}{16}\delta^{mnp}_{i\,j\,k}\, T^{vw}_{rstu}\, T^{rstu}_{vw} &= 0 \,, \end{aligned} \tag{3.13}$$

where in the last identity the antisymmetrization does not include the indices v, w.

The above considerations apply also to other dimensions. Observe that the T-tensor transforms according to a tensor product of the representation associated with the field strengths, in this case the **56** of $\mathrm{E}_{7(7)}$, with the adjoint representation of G, in this case the **133** of $\mathrm{E}_{7(7)}$. Likewise, the T-tensor for $d=7$ belongs to the $\mathbf{10}\times\mathbf{24}$ representation of SL(5), for $d=6$ it belongs to the $\mathbf{16}\times\mathbf{45}$ representation of $\mathrm{SO}(5,5)$, and for $d=5$ to the $\mathbf{27}\times\mathbf{78}$ representation of $\mathrm{E}_{6(6)}$. Obviously, these representations are all reducible and, as we shall discuss in the next section, a consistent gauging requires the T-tensor to take its values in a smaller representation.

Before completing the analysis leading to a consistent gauging we stress that all variations of the Lagrangian are expressed entirely in terms of the T-tensor, as its variations is again proportional to the same tensor. This includes the SU(8) covariant derivative of the T-tensor, which follows directly from (3.10) upon the substitutions $\delta\to D_\mu$ and $\Sigma\to\mathcal{P}_\mu$. A viable gauging requires that the T-tensor satisfies a number of rather nontrivial identities, as we will discuss shortly, but the new terms in the Lagrangian and transformation rules have a universal form, irrespective of the gauge group (except from the term (2.23), which must be included depending on the nature of the gauge group). Let us first describe these new terms. First of all, the order-g variations from Cartan-Maurer equations are cancelled by the variations of new masslike terms for the fermions,

$$\begin{aligned}\mathcal{L}_{\rm masslike} \;=\;& g\,e\Big\{\tfrac{1}{2}\sqrt{2}\,A_{1ij}\overline{\psi}{}^i_\mu\gamma^{\mu\nu}\psi^j_\nu+\tfrac{1}{6}A_{2i}^{jkl}\,\overline{\psi}{}^i_\mu\gamma^\mu\chi_{jkl}\\ &+A_3^{ijk,lmn}\,\overline{\chi}_{ijk}\chi_{lmn}+\text{h.c.}\Big\}\,,\end{aligned}\tag{3.14}$$

and by new terms in the supersymmetry transformations of the fermion fields,

$$\begin{aligned}\delta_g\bar{\psi}{}^i_\mu &= -\sqrt{2}g\,A_1^{ij}\,\bar{\epsilon}_j\gamma_\mu\,,\\ \delta_g\chi^{ijk} &= -2g\,A_{2l}{}^{ijk}\,\bar{\epsilon}^l\,.\end{aligned}\tag{3.15}$$

Finally at order g^2, supersymmetry requires a potential for the spinless fields,

$$P(\mathcal{V})=g^2\Big\{\tfrac{1}{24}|A_{2i}{}^{jkl}|^2-\tfrac{1}{3}|A_1^{ij}|^2\Big\}\,.\tag{3.16}$$

In passing we note that the SU(8) covariant derivative of A_1 is proportional to A_2,

$$D_\mu A_1^{ij}=\tfrac{1}{12}\sqrt{2}\,A_{2\,klm}^{(i}\,\mathcal{P}_\mu^{j)klm}\,.\tag{3.17}$$

A similar, but slightly more complicated result holds for the derivative of A_2.

The new terms in the Lagrangian and transformation rules have a form that does not depend on the details of the gauging. Note that the tensors A_1^{ij}, $A_{2i}{}^{jkl}$ and $A_3^{ijk,lmn}$ have certain symmetry properties dictated by the way they appear in the Lagrangian (3.14): A_1 is symmetric in (ij), A_2 is fully antisymmetric in $[jkl]$ and A_3 is antisymmetric in $[ijk]$ as well as in $[lmn]$ and symmetric under the interchange $[ijk]\leftrightarrow[lmn]$. Therefore

d	$\mathrm{H_R}$	A_1	A_2	A_3
7	USp(4)	$\mathbf{1}+\mathbf{5}$	$\mathbf{5}+\mathbf{10}+\mathbf{14}+\mathbf{35}$	$\mathbf{1}+\mathbf{5}+\mathbf{14}+\mathbf{30}+\mathbf{35}+\mathbf{35}'$
6	USp(4)×USp(4)	$(\mathbf{4},\mathbf{4})$	$(\mathbf{4},\mathbf{4})+(\mathbf{4},\mathbf{4})$ $+(\mathbf{4},\mathbf{16})+(\mathbf{16},\mathbf{4})$	$(\mathbf{4},\mathbf{4})+(\mathbf{4},\mathbf{16})+(\mathbf{16},\mathbf{4})$ $+(\mathbf{16},\mathbf{16})$
5	USp(8)	$\mathbf{36}$	$\mathbf{27}+\mathbf{42}+\mathbf{315}$	$\mathbf{1}+\mathbf{27}+\mathbf{36}+\mathbf{308}$ $+\mathbf{315}+\mathbf{792}+\mathbf{825}$
4	SU(8)	$\mathbf{36}+\overline{\mathbf{36}}$	$\mathbf{28}+\overline{\mathbf{28}}+\mathbf{420}+\overline{\mathbf{420}}$	$\mathbf{420}+\overline{\mathbf{420}}+\mathbf{1176}+\overline{\mathbf{1176}}$
3	SO(16)	$\mathbf{1}+\mathbf{135}$	$\mathbf{128}+\overline{\mathbf{1920}}$	$\mathbf{1}+\mathbf{1820}+\overline{\mathbf{6435}}$

Table 3: Possible fermion mass terms for maximal supergravities in various dimensions assigned to irreducible R-symmetry representations. Note that in $d=7$ dimensions the tensors A_1 and A_3 are antisymmetric in the fermion indices.

these tensors transform under SU(8) according to the representations

$$\begin{aligned} A_1 &: \mathbf{36}+\overline{\mathbf{36}}\,, \\ A_2 &: \mathbf{28}+\overline{\mathbf{28}}+\mathbf{420}+\overline{\mathbf{420}}\,, \\ A_3 &: \mathbf{420}+\overline{\mathbf{420}}+\mathbf{1176}+\overline{\mathbf{1176}}\,. \end{aligned} \tag{3.18}$$

This analysis can be repeated for each of the maximal supergravities. In table 3 we indicate all possible masslike terms for the fermions by indicating the irreducible representations of $\mathrm{H_R}$ to which they belong. These results are relevant for deducing the constraints on the T-tensor in the next section. We observe here that only in $d=7$ dimensions the matrices A_1 and A_3 are antisymmetric, due to the property of pseudo Majorana spinors.

Because covariant variations of the T-tensor are again proportional to the T-tensor, and the potential is SU(8) invariant, variations of the potential are quadratic in the T-tensor. Stationary points of the potential are subject to the condition that Ω^{ijkl} must be anti-selfdual, where the tensor Ω is defined by [22],

$$\Omega^{ijkl} = \tfrac{3}{4} A_{2\,m}^{n[ij}\, A_{2\,n}^{kl]m} - A_1^{m[i}\, A_{2\,m}^{jkl]}\,. \tag{3.19}$$

Expanding the potential about a stationary point leads to a mass term, which is again quadratic in the T-tensor. We refer to the explicit expressions given in [22]. All this is completely generic and similar formulae can be derived for maximal supergravity in any spacetime dimension.

The mass term for the vector fields is generated by the gauge covariantizations in the scalar kinetic term. Making use of (3.4), one arrives immediately at the following relation

$$\mathcal{L} = -\tfrac{1}{12} e\,|\mathcal{P}_\mu^{ijkl}|^2 \longrightarrow -\tfrac{1}{12} e g^2\, A_\mu^{AB}\, A^{\mu\,CD}\, \mathcal{P}_{AB\,ijkl}\, \mathcal{P}_{CD}^{ijkl}\,. \tag{3.20}$$

The physical mass is, however, given in terms of the square of the T-tensor as one notes after combining the mass term with the kinetic terms for the vector fields,

$$
\begin{aligned}
\mathcal{L} \;=\; & -\tfrac{1}{8}e\,\left[F_{\mu\nu}^{AB}[(u+v)^{-1}]^{AB}{}_{ij}\right]\,\left[F^{\mu\nu\,CD}[(u+v)^{-1}]_{CD}{}^{kl}\right]\,\delta^{ij}_{kl} \\
& -\tfrac{1}{3}e\,g^2\,\left[A_{\mu}^{AB}[(u+v)^{-1}]^{AB}{}_{ij}\right]\,\left[A^{\mu\,CD}[(u+v)^{-1}]_{CD}{}^{kl}\right]\,T^{ij}_{mnpq}\,T^{mnpq}_{kl}\,, \qquad (3.21)
\end{aligned}
$$

where we suppressed the terms proportional to $\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^{AB}F_{\rho\sigma}^{CD}$ as well as the terms (2.23).

4 Group-theoretic analysis

The three SU(8) covariant tensors, A_1, A_2 and A_3, which depend only on the spinless fields, must be linearly related to the T-tensor, because they were introduced for the purpose of cancelling the variations proportional to the T-tensors. To see how this can be the case, let us analyze the SU(8) content of the T-tensor. As we mentioned already, the T-tensor is cubic in the 56-bein, and as such it constitutes a tensor that transforms under $\mathrm{E}_{7(7)}$. The transformation properties were given in (3.10), where we made use of the fact that the T-tensor consists of a product of the fundamental times the adjoint representation of $\mathrm{E}_{7(7)}$. Hence the T-tensor comprises the representations,

$$
\mathbf{56}\times\mathbf{133} = \mathbf{56}+\mathbf{912}+\mathbf{6480}\,. \qquad (4.1)
$$

The representations on the right-hand side can be decomposed under the action of SU(8),

$$
\begin{aligned}
\mathbf{56} &= \mathbf{28}+\overline{\mathbf{28}}\,, \\
\mathbf{912} &= \mathbf{36}+\overline{\mathbf{36}}+\mathbf{420}+\overline{\mathbf{420}}\,, \\
\mathbf{6480} &= \mathbf{28}+\overline{\mathbf{28}}+\mathbf{420}+\overline{\mathbf{420}}+\mathbf{1280}+\overline{\mathbf{1280}}+\mathbf{1512}+\overline{\mathbf{1512}}\,. \qquad (4.2)
\end{aligned}
$$

These representations should correspond to the SU(8) representations to which the tensors A_1-A_3 (and their complex conjugates) belong. However, there is a mismatch between (4.2) and (3.18), which shows that the T-tensor is constrained. In view of (3.10) this constraint should amount to suppressing complete representations of $\mathrm{E}_{7(7)}$ in order that its variations and derivatives remain consistent. Therefore the conclusion is that the T-tensor cannot contain the **6480** representation of $\mathrm{E}_{7(7)}$, so that it consists at most of the $\mathbf{28}+\mathbf{36}+\mathbf{420}$ representation of SU(8) (and its complex conjugate). This implies that A_3 is not an independent tensor and can be expressed in terms of A_2 and that the T-tensor is decomposable into A_1 and A_2. Indeed this was found by explicit calculation, which reveals the relations

$$
\begin{aligned}
T_i^{jkl} &= -\tfrac{3}{4}A_{2i}{}^{jkl}+\tfrac{3}{2}A_1^{j[k}\,\delta_i^{l]}\,, \\
T_{ijkl}^{mn} &= -\tfrac{4}{3}\delta_{[i}^{[m}T^{n]}_{jkl]}\,, \\
A_3^{ijk,lmn} &= -\tfrac{1}{108}\sqrt{2}\,\varepsilon^{ijkpqr[lm}T^{n]}_{pqr}\,. \qquad (4.3)
\end{aligned}
$$

d	G	T
7	SL(5)	$\mathbf{10} \times \mathbf{24} = \mathbf{10} + \mathbf{15} + \mathbf{40} + \mathbf{175}$
6	SO(5,5)	$\mathbf{16} \times \mathbf{45} = \mathbf{16} + \mathbf{144} + \mathbf{560}$
5	$\mathrm{E}_{6(6)}$	$\mathbf{27} \times \mathbf{78} = \mathbf{27} + \mathbf{351} + \mathbf{1728}$
4	$\mathrm{E}_{7(7)}$	$\mathbf{56} \times \mathbf{133} = \mathbf{56} + \mathbf{912} + \mathbf{6480}$
3	$\mathrm{E}_{8(8)}$	$\mathbf{248} \times \mathbf{248} = \mathbf{1} + \mathbf{248} + \mathbf{3875} + \mathbf{27000} + \mathbf{30380}$

Table 4: Decomposition of the T-tensor in various dimensions for maximal supergravities in terms of irreducible representations of G.

Observe that the first equation implies that the **28** representation is also suppressed, because the combination of $T_i^{ikl} = 0$ with this equation implies that

$$T_i^{[ijk]} = 0\,. \tag{4.4}$$

Hence the T-tensor transforms under $\mathrm{E}_{7(7)}$ according to the **912** representation which decomposes into the **36** and **420** representations of SU(8) and their complex conjugates residing in the tensors A_1 and A_2, respectively,

$$A_1^{ij} = \tfrac{4}{21} T_k^{ikj}\,, \qquad A_{2i}^{jkl} = -\tfrac{4}{3} T_i^{[jkl]}\,. \tag{4.5}$$

The fact that the T-tensor is restricted to a particular representation of $\mathrm{E}_{7(7)}$ ensures that the identities (3.13) quadratic in the T-tensor suffice to cancel the order g^2 variations in the Lagrangian. Therefore the condition that the T-tensor belongs to the **912** representation is a sufficient criterion for establishing the viability of a given gauging.

We concentrated on the $d = 4$ theory, but many of the above features are generic and apply in other dimensions, where the complications related to electric-magnetic duality are absent. For instance, we show the representations of the unrestricted T-tensors for $3 \leq d \leq 7$ spacetime dimensions in table 4.[3]

In $d = 3$ dimensions the $\mathbf{248} + \mathbf{30380}$ representation should be suppressed as it is antisymmetric and cannot appear in the Chern-Simons coupling. Just as above, comparing the R-symmetry representations contained in the G-representations listed in table 4, to the R-symmetry representations for the masslike terms listed in table 3, shows that the following representations cannot appear: for $d = 7$ the representation **175**, for $d = 6$ the **560** representation, for $d = 5$ the **1728** representation, for $d = 4$ the **6480** representation and for $d = 3$ the **27000** representation. Following the arguments presented above, we establish the following representation content for the T-tensors

[3] The $d = 3$ theory has initially no vector fields, but those can be included by adding Chern-Simons terms. These gauge fields are used to gauge some of the $\mathrm{E}_{8(8)}$ isometries [11].

and their branching into R-symmetry representations:

$$\begin{aligned}
d &= 7 \quad : \quad \mathbf{15} \rightarrow \mathbf{1} + \mathbf{14}\,, \\
d &= 6 \quad : \quad \mathbf{144} \rightarrow (\mathbf{4},\mathbf{4}) + (\mathbf{4},\mathbf{16}) + (\mathbf{16},\mathbf{4})\,, \\
d &= 5 \quad : \quad \mathbf{351} \rightarrow \mathbf{36} + \mathbf{315}\,, \\
d &= 4 \quad : \quad \mathbf{912} \rightarrow \mathbf{36} + \overline{\mathbf{36}} + \mathbf{420} + \overline{\mathbf{420}}\,, \\
d &= 3 \quad : \quad \mathbf{1} + \mathbf{3875} \rightarrow \mathbf{1} + \mathbf{135} + \mathbf{1820} + \overline{\mathbf{1920}}\,.
\end{aligned} \tag{4.6}$$

These constraints on the T-tensor must be satisfied for any gauging. All the corresponding R-symmetry representations must appear in the Lagrangian with the right multiplicity. It turns out that, with the exception of $d = 3$, all representations are covered by the tensors A_1 and A_2, so that the tensor A_3 is not independent. For $d = 3$, the tensor A_3 contains the **1820** representation of SO(16), which is not present in A_1 and A_2. Therefore A_3 is independent in this case.

It is possible to rephrase some of the above in the following way. A gauging is characterized by a real *embedding matrix* $\Theta_M{}^\alpha$ which defines how the gauge group is embedded into $\mathrm{E}_{7(7)}$. Here the indices α and M belong to the adjoint and the fundamental representation of $\mathrm{E}_{7(7)}$, respectively. Hence, $\alpha = 1, \ldots, 133$, and $M = 1, \ldots, 56$. The gauge group generators can now be labeled by indices belonging to the fundamental representation,

$$t_M = \Theta_M{}^\alpha\, t_\alpha\,, \tag{4.7}$$

where the t_α are the 133 generators of $\mathrm{E}_{7(7)}$ in an arbitrary representation. The fact that the t_M generate a group, implies that the embedding matrix satisfies the condition,

$$\Theta_M{}^\alpha\, \Theta_N{}^\beta\, f_{\alpha\beta}{}^\gamma = f_{MN}{}^P\, \Theta_P{}^\gamma\,, \tag{4.8}$$

where the $f_{\alpha\beta}{}^\gamma$ and $f_{MN}{}^P$ are the structure constants of $\mathrm{E}_{7(7)}$ and the gauge group, respectively. This condition implies that the embedding matrix is invariant under the gauge group. In principle one is dealing with 28 electric and 28 magnetic charges, which transform according to the **56** representation. Only the electric charges can couple locally and the subset of the corresponding generators that are involved in this gauging were previously denoted by t_{AB}. As previously stated, their embedding in the **56** is constrained by the condition (2.20), namely that they should be contained in the global symmetry group $\mathrm{G_e}$ of the Lagrangian or, in other words, that they correspond to *electric* charges. The embedding matrix is in fact directly related to the T-tensor, which transforms under $\mathrm{E}_{7(7)}$ according to the tensor product $\mathbf{56} \times \mathbf{133}$. Namely, by making a field-dependent $\mathrm{E}_{7(7)}$ transformation the matrix $\mathcal{V}$ can be reduced to the identity, so that the T-tensor is field independent and equal to the generators t_{AB} appropriately contracted by E. Recalling that the indices of the **56** refer to antisymmetric index pairs that are denoted by lower or upper indices, we find for $M = [IJ]$ with lower indices,

$$T_M{}^\alpha\, t_\alpha\Big|_0 = \Theta_M{}^\alpha\, t_\alpha = \tfrac{1}{2}(\mathsf{U}_{IJ}{}^{AB} + \mathsf{V}_{IJAB})\, t_{AB}\,, \tag{4.9}$$

where here and henceforth the t_α are the $\mathrm{E}_{7(7)}$ generators in the fundamental representation. The complex conjugate of this relation refers to the embedding matrix with M referring to the upper indices $[IJ]$. The T-tensor is obtained by transforming $T|_0$ under $\mathrm{E}_{7(7)}$ with a field dependent transformation defined by $\mathcal{V}(x)$:

$$T_M{}^\alpha[\Theta]\, t_\alpha = \mathcal{V}^{-1}_M{}^N\, \Theta_N{}^\alpha\, \mathcal{V}^{-1} t_\alpha \mathcal{V}\,. \tag{4.10}$$

As a final condition, which was explained above, the T-tensor is restricted to be contained in the **912** representation. Therefore the embedding matrix has to belong to that representation. Defining appropriate projection operators, this condition can be expressed as follows,

$$\mathbb{P}_{(912)M}{}^{\alpha\,N}{}_\beta\, \Theta_N{}^\beta \;=\; \Theta_M{}^\alpha\,, \tag{4.11}$$

where $\mathbb{P}_{(912)}$ is the projection operator on the **912** representation in the product space (4.1). As we explain in detail in the appendix, this projector may be explicitly written in terms of the generators t_α as

$$\begin{aligned}
\mathbb{P}_{(912)M}{}^{\alpha\,N}{}_\beta &= -\tfrac{12}{7}\,(t^\alpha)_K{}^N\,(t_\beta)_M{}^K + \tfrac{4}{7}(t^\alpha)_M{}^K\,(t_\beta)_K{}^N + \tfrac{1}{7}\,\delta_M{}^N\delta^\alpha{}_\beta\,,\\
\mathbb{P}_{(6480)M}{}^{\alpha\,N}{}_\beta &= \tfrac{12}{7}\,(t^\alpha)_K{}^N\,(t_\beta)_M{}^K - \tfrac{132}{133}(t^\alpha)_M{}^K\,(t_\beta)_K{}^N + \tfrac{6}{7}\,\delta_M{}^N\delta^\alpha{}_\beta\,,\\
\mathbb{P}_{(56)M}{}^{\alpha\,N}{}_\beta &= \tfrac{8}{19}\,(t^\alpha)_M{}^K\,(t_\beta)_K{}^N\,,
\end{aligned} \tag{4.12}$$

where indices α,β are raised and lowered with the invariant metric $\eta_{\alpha\beta} = \mathrm{Tr}(t_\alpha t_\beta)$.

Similarly, in five dimensions the projectors on the **351**, **1728** and the **27** representations appearing in the branching rules shown in table 4 are obtained from (A.5)

$$\begin{aligned}
\mathbb{P}_{(351)\Lambda}{}^{a\,\Sigma}{}_b &= -\tfrac{6}{5}\,(t^a)_\Gamma{}^\Sigma\,(t_b)_\Lambda{}^\Gamma + \tfrac{3}{10}(t^a)_\Lambda{}^\Gamma\,(t_b)_\Gamma{}^\Sigma + \tfrac{1}{5}\,\delta_\Lambda{}^\Sigma\delta^a{}_b\,,\\
\mathbb{P}_{(1728)\Lambda}{}^{a\,\Sigma}{}_b &= \tfrac{6}{5}\,(t^a)_\Gamma{}^\Sigma\,(t_b)_\Lambda{}^\Gamma - \tfrac{42}{65}(t^a)_\Lambda{}^\Gamma\,(t_b)_\Gamma{}^\Sigma + \tfrac{4}{5}\,\delta_\Lambda{}^\Sigma\delta^a{}_b\,,\\
\mathbb{P}_{(27)\Lambda}{}^{a\,\Sigma}{}_b &= \tfrac{9}{26}\,(t^a)_\Lambda{}^\Gamma\,(t_b)_\Gamma{}^\Sigma\,,
\end{aligned} \tag{4.13}$$

where Λ, Γ, $\Sigma = 1,\ldots,27$ and a, $b = 1,\ldots,78$ label the basis elements in the fundamental and the adjoint representation of $\mathrm{E}_{6(6)}$, respectively. The projector $\mathbb{P}_{(351)}$ on the space of the embedding matrices $\Theta_\Lambda{}^a$ will be used to implement the supersymmetry restriction on the T-tensor and thus to define the allowed gaugings.

As we shall see in the sequel, when we consider the four-dimensional gaugings in the $\mathrm{E}_{6(6)}$-basis, the **351** representation can be obtained from the branching of the **912** of $\mathrm{E}_{7(7)}$ with respect to the $\mathrm{E}_{6(6)}$ subgroup and the corresponding projector in the space of the embedding matrices can be derived as a suitable restriction of $\mathbb{P}_{(912)}$.

The above projection operators are used in the computer-aided analysis of the T-tensor that we will make use of in later sections.

5 Scherk-Schwarz reductions

In this section we summarize a number of features related to the modified dimensional reduction scheme proposed by Scherk and Schwarz [6]. This scheme applies to a theory in higher dimensions with a rigid internal symmetry group G. It consists of an ordinary dimensional reduction on a hypertorus, where an extra dependence on the torus coordinates is introduced by applying a finite, uniform G-transformation that depends nontrivially on the these coordinates. Subsequently one retains only the lowest Fourier components. Because of the invariance the internal symmetry transformation cancels out up to those terms in the Lagrangian that contain derivatives with respect to the torus coordinates. As one can easily verify, this reduction defines a consistent truncation of the higher-dimensional theory, which thus corresponds to a deformation of the lower-dimensional theory that one obtains by ordinary dimensional reduction. The dependence of the internal symmetry transformation deserves some comment. When compactifying on a hypertorus T^n with coordinates y^m, one specifies an n-dimensional abelian subgroup of the internal symmetry group G, with generators t_m so that the group element $\hat{g}(y) \in$ G equals $\hat{g}(y) = \exp[g\, y^m\, t_m]$. For convenience we have introduced a coupling constant g. The deformation of the lower-dimensional theory is governed by the Lie-algebra valued quantities $\hat{g}^{-1}\partial_m \hat{g}$ which are obviously y-independent and equal to the matrices $g\, t_m$. The deformed theory is always related to a possible gauging of the theory, with the parameter g playing the role of the gauge coupling constant, because the graviphotons couple to the y-dependent quantities. The t_m are the charges that couple to the graviphotons according to the description given in earlier sections. The other gauge fields that participate in the gauging are the gauge fields that already exist in higher dimensions (or possibly, tensor fields that give rise to vectors in lower dimensions). Because these fields transform under the internal symmetry group in higher dimensions, they will generically be charged with respect to the graviphotons. For maximal supergravity, the gaugings define the only known supersymmetric deformations. In this paper we start from the other end; we analyze possible gaugings in lower dimensions and identify some of them afterwards with the result of a Scherk-Schwarz reduction. The inequivalent Scherk-Schwarz gaugings correspond to the different conjugacy classes of $\hat{g}$.

In order to properly identify a Scherk-Schwarz gauging we consider a generic Lagrangian of gravity in $d + n$ spacetime dimensions, coupled to scalars, vectors and higher-rank antisymmetric gauge fields. The Lagrangian depends only on Newton's constant and there are no other dimensionfull coupling constants. In addition, the Lagrangian is invariant under a group G and the scalar fields are assumed to parametrize a homogeneous target space. Hence we consider a typical Lagrangian with an Einstein-

Hilbert term, scalar fields and a k-rank gauge field, with possible interactions,

$$\mathcal{L}_{d+n} = -\frac{1}{2\,\kappa_{d+n}^2} E \left[R + \mathcal{P}_M^2 + \frac{(k+1)^2}{k!} \Big(\partial_{[M_1} A_{M_2 \cdots M_{k+1}]} \Big)^2 + \cdots \right] , \tag{5.1}$$

as well as a single coupling constant κ_{d+n}^2. The kinetic term for the tensor gauge field may be modified by terms depending on the scalar fields, but this dependence has been suppressed. For convenience of notation, only in this section the indices $M, N \ldots$ and $A, B \ldots$ denote $(d+n)$-dimensional world and target indices; the world and target indices of the torus are $m, n \ldots$ and $a, b \ldots$ while those of the d-dimensional space-time are $\mu, \nu \ldots$ and $\alpha, \beta \ldots$. The vielbein field is denoted by $E_M{}^A$ and $E = \det(E_M{}^A)$. We assume that the Lagrangian remains unchanged under a simultaneous rescaling of the coupling constant and the fields,

$$\begin{array}{rclcrcl} E_M{}^A & \to & \mathrm{e}^{-\alpha} E_M{}^A \,, & \qquad & A_{M_1 \cdots M_k} & \to & \mathrm{e}^{-k\alpha} A_{M_1 \cdots M_k} \,, \\ \mathcal{P}_M & \to & \mathcal{P}_M \,, & & \kappa_{d+n}^2 & \to & \mathrm{e}^{(2-d-n)\alpha}\, \kappa_{d+n}^2 \,. \end{array} \tag{5.2}$$

Note that the tangent space tensors corresponding to $\mathcal{P}_M$ and the field strengths (which may include modifications with scalar fields), $F_{M_1 \cdots M_{k+1}} = (k+1)\,\partial_{[M_1} A_{M_2 \cdots M_{k+1}]}$, scale with the same factor $\exp[\alpha]$. The composite connection field $\mathcal{Q}_M$, which does not appear explicitly in the above formula, scales precisely as $\mathcal{P}_M$. All supergravity Lagrangians that follow from 11-dimensional supergravity by standard dimensional reduction, are of this type.

Under standard compactification on a torus T^n the Lagrangian that describes the massless modes in d spacetime dimensions remains invariant under the symmetry group G of the original $(d+n)$-dimensional theory, and under GL(n). While the transformations under the SL(n) subgroup are obvious from the index structure of the various fields after dimensional reduction, this is not so for the scale transformation. These scale transformations will be denoted by SO$(1,1)$ in the next sections. Although the symmetries are not preserved by the modified dimensional reduction, one can still classify the various fields with respect to these transformations. They originate from a combined uniform rescaling of the torus coordinates and (5.2), such that the coupling constant κ_d^2 in d dimensions remains constant. The latter coupling constant is inversely proportional to the torus volume.

Upon dimensional reduction the vielbein field decomposes into the d-dimensional vielbein field $e_\mu{}^\alpha$, n -photon fields B_μ^m, a graviscalar ϕ and an internal vielbein field $\hat{e}_m{}^a$ with $\det[\hat{e}] = 1$. The latter parametrizes a SL(n)/SO(n) target space. In order to re-obtain the Einstein-Hilbert Lagrangian in d dimensions, one performs a Weyl rescaling of the vielbein $e_\mu{}^\alpha \to \exp[-n\phi/(d-2)]\, e_\mu{}^\alpha$. The tensor field decomposes into tensors that transform irreducibly with respect to both the d-dimensional Lorentz transformations and SL(n). The tensor fields are in general not redefined by Weyl rescalings as this would affect the form of their gauge transformations. These steps are rather standard and we refrain from giving further details. After they have been carried out, the fields $e_\mu{}^\alpha$, $\hat{e}_m{}^a$, $\mathcal{P}_\mu$ and $\mathcal{Q}_\mu$ remain unaffected by the scale transformation, while the remaining fields scale as follows,

$$\begin{aligned} \mathrm{e}^{\phi} &\rightarrow \mathrm{e}^{(d-2)\alpha/n}\, \mathrm{e}^{\phi}\,, \\ B_\mu^m &\rightarrow \mathrm{e}^{-[(d-2)/n+1]\alpha}\, B_\mu^m\,, \\ A_{\mu_1\cdots\mu_k} &\rightarrow \mathrm{e}^{-k\,\alpha}\, A_{\mu_1\cdots\mu_k}\,, \\ A_{m_1\cdots m_p\,\mu_{p+1}\cdots\mu_k} &\rightarrow \mathrm{e}^{[(d-2)p/n+p-k]\alpha}\, A_{m_1\cdots m_p\,\mu_{p+1}\cdots\mu_k}\,, \\ A_{m_1\cdots m_k} &\rightarrow \mathrm{e}^{k(d-2)\alpha/n}\, A_{m_1\cdots m_k}\,. \end{aligned} \tag{5.3}$$

In the Lagrangian the invariance under the scale transformations reflects itself in certain exponential factors of the graviscalar. In standard dimensional reduction, derivatives with respect to the torus coordinates vanish and the scale invariance is exact. In the Scherk-Schwarz scheme these derivatives no longer vanish and give rise to mass terms and a potential which break this invariance. The same phenomenon is encountered when one retains the massive modes in standard dimensional reduction where the Kaluza-Klein masses and charges break the scale invariance. These masses and charges are inversely proportional to the periodicity lengths associated with the torus. However, when one combines the scale transformations with a simultaneous, uniform rescaling of the torus (and thus of the Kaluza-Klein masses and charges) then the Lagrangian remains unchanged. For the Scherk-Schwarz reduction one has the same situation. The group element $\hat{g}(y)$ is not invariant under a uniform scaling of the y^m, but we can simultaneously scale the coupling constant g such that $g\,y^m$ remains invariant. Consequently, the Lagrangian is not invariant under the scale transformations for fixed g, but it remains unchanged provided we scale g as well. In this way, the lack of scale invariance is precisely characterized by the corresponding power of g.

Suppressing the higher Fourier modes and assigning the following scale transformation to the coupling constant g,

$$g \rightarrow \mathrm{e}^{[(d-2)/n+1]\,\alpha}\, g\,, \tag{5.4}$$

the Lagrangian of the reduced theory remains unchanged under the combined scale transformations. Specifically, the following quantities which involve derivatives ∂_m, scale consistently under uniform scale transformations of the torus coordinates, *i.e.*,

$$\begin{aligned} \mathcal{P}_m &\rightarrow \mathrm{e}^{[(d-2)/n+1]\alpha}\, \mathcal{P}_m\,, \\ \mathcal{Q}_m &\rightarrow \mathrm{e}^{[(d-2)/n+1]\alpha}\, \mathcal{Q}_m\,, \\ F_{m_1\cdots n_p\,\mu_{p+1}\cdots\mu_{k+1}} &\rightarrow \mathrm{e}^{[(d-2)p/n+p-k]\alpha}\, F_{m_1\cdots n_p\,\mu_{p+1}\cdots\mu_{k+1}}\,, \end{aligned} \tag{5.5}$$

by virtue of the scale transformation of the coupling constant. Note that the field strengths will in general be modified by certain functions of the scalar fields (originating from the original $(d+n)$-dimensional theory), but those modifications do not affect the scaling behavior. With the above transformation rules, the (non-negative) potential,

$$P = \frac{e}{2\,\kappa_d^2}\left[\left(\mathrm{e}^{-[1+n/(d-2)]\phi}\,\mathcal{P}_m\right)^2 + \frac{1}{k!}\left(\mathrm{e}^{-[k+1+n/(d-2)]\phi}\,F_{m_1\cdots m_{k+1}}\right)^2\right]\,, \tag{5.6}$$

remains unaffected. For finite values of ϕ these potentials can only have stationary points when $\mathcal{P}_m$ and $F_{m_1\cdots m_{k+1}}$ vanish. These stationary points have zero cosmological constant. When ϕ tends to infinity, the potential will vanish as well. We return to the the properties of the Lagrangian at the end of this section.

The terms in parentheses can be interpreted as components of the T-tensors, multiplied with the coupling constant g. Therefore it follows that the T-tensors scale according to

$$T \rightarrow \mathrm{e}^{-[(d-2)/n+1]\,\alpha}\,T\,, \tag{5.7}$$

so that $g\,T$ remains invariant. Likewise also $g\,B^m{}_\mu$ remains invariant. We can be a little more precise here, by also considering the fermions and following what happens to the transformation rules under the reduction. After proper Weyl rescalings and rediagonalization, one then concludes that the Scherk-Schwarz T-tensors decompose as follows,

$$\begin{aligned} g\,A_1 &= \mathrm{e}^{-[1+n/(d-2)]\phi}\Big[\mathcal{Q}_m \oplus \mathrm{e}^{-k\phi}F_{m_1\cdots m_{k+1}}\Big]\,, \\ g\,A_2 &= \mathrm{e}^{-[1+n/(d-2)]\phi}\Big[\mathcal{Q}_m \oplus \mathcal{P}_m \oplus \mathrm{e}^{-k\phi}F_{m_1\cdots m_{k+1}}\Big]\,. \end{aligned} \tag{5.8}$$

Hence, one identifies the same contributions to the T-tensor as in (5.6). There are some noteworthy features. One is that the contributions of $\mathcal{Q}_m$, which contributes to both A_1 and A_2, cancel in the potential. The other is that the (negative) contribution from A_1 to the potential is always compensated by the (positive) contribution from A_2 in order that the final result remains non-negative. The contributions from the tensor fluxes can only be present for multiple Scherk-Schwarz reductions where $n \geq k+1$. We conclude that the Scherk-Schwarz T-tensors have a uniform behavior under $\mathrm{SO}(1,1)$, with a weight given by (5.7). Furthermore they transform as tensors under $\mathrm{SL}(n)$. We intend to exhibit a number of examples of gaugings corresponding to hitherto new Scherk-Schwarz reductions in [13].

We already observed that the potential has only stationary points when $\mathcal{P}_m = F_{m_1\cdots m_{k+1}} = 0$. Obviously, for large and positive values of ϕ the potential tends to zero. In order to have $\mathcal{P}_m = 0$, there are restrictions on the generators t_m involved in the Scherk-Schwarz reduction. Namely, at the stationary point, the coset representative $\mathcal{V}_0$ and the generators t_m should satisfy the condition that $\mathcal{V}_0^{-1}t_m\,\mathcal{V}_0$ belongs to the

Lie algebra associated with the isotropy group H. Since they thus belong to the same conjugacy class, we can restrict the t_m to a Cartan subalgebra associated with H, so that the t_m are anti-hermitean. From this observation we derive the condition,

$$\mathcal{V}_0\mathcal{V}_0^\dagger\, t_m = t_m\, \mathcal{V}_0\mathcal{V}_0^\dagger\,. \tag{5.9}$$

Hence, we have mutually commuting generators t_m which all commute with $\mathcal{V}_0\mathcal{V}_0^\dagger$. As the latter is a positive-definite hermitean matrix, it can be diagonalized with positive eigenvalues and the t_m can simultaneously be diagonalized. Furthermore, there is a gauge in which the coset representative $\mathcal{V}$ is hermitean, from which we deduce that the stationary point is in fact part of a zero-potential valley, spanned by the scalars that are invariant under the group generated by the t_m. The fields that are not invariant under this group, must therefore vanish at the stationary point. Of course, whether or not the field strengths $F_{m_1\cdots m_{k+1}}$ vanish is a question that is independent from the above considerations.

6 The known gaugings in $d = 4$ dimensions

In this section we demonstrate how the group-theoretical approach of this paper allows us to straightforwardly establish the viability of the known gaugings of maximal supergravity in 4 dimensions. The strategy is to make an assumption about the gauge group G_g, or about the electric subgroup $\mathrm{G_e}$ which contains G_g as a subgroup, and then analyze the constraint (4.11). We do this by comparing the branching of the $\mathbf{133}\times\mathbf{56}$ representation under a given electric subgroup $\mathrm{G_e}$, to the representations that are allowed for the T-tensor according to (4.11). In this way we can identify common representations in the product representation and ultimately determine the possible assignment of the T-tensor to representations of $\mathrm{G_e}$. At the same time we deduce which gauge fields and $\mathrm{E}_{7(7)}$ generators are involved in the corresponding gauging. In a sequel we review the situation regarding the known gaugings based on $\mathrm{G_e}$ equal to $\mathrm{SL}(8,\mathbb{R})$, $\mathrm{E}_{6(6)}\times\mathrm{SO}(1,1)$ and $\mathrm{SU}^\star(8)$.

In the comparison and the identification of the various representations, the projector $\mathbb{P}_{(912)}$, defined in (4.12), plays a central role. The part of the analysis that involves its explicit form is done by means of the computer. Because a direct handling of the $(56\cdot 133)\times(56\cdot 133)$ matrix $\mathbb{P}_{(912)}$ is rather cumbersome, we construct an orthogonal basis of 912 embedding matrices spanning the $\mathbf{912}$ representation. Putting some of the representations in the $\mathbf{56}$ and the $\mathbf{133}$ representations to zero, we analyze a system of corresponding linear equations which then reveals the precise branching rules.

We should stress that this approach is by no means limited to $d = 4$ and can also be applied to gauged supergravity theories in other dimensions. However, the analysis is most subtle for $d = 4$ in view of electric-magnetic duality and, in this section we restrict ourselves to that case. To demonstrate our strategy, we will now proceed and discuss the three classes of known gaugings.

6.1 Gauge groups embedded in $\mathrm{SL}(8,\mathbb{R})$

The first class of gaugings concerns gauge groups that can be embedded into the $\mathrm{SL}(8,\mathbb{R})$ subgroup of $\mathrm{E}_{7(7)}$. The relevant branching rules into $\mathrm{SL}(8,\mathbb{R})$ representations are,

$$\begin{aligned} \mathbf{56} &\rightarrow \mathbf{28} + \overline{\mathbf{28}}\,, \\ \mathbf{133} &\rightarrow \mathbf{63} + \mathbf{70}\,, \\ \mathbf{912} &\rightarrow \mathbf{36} + \mathbf{420} + \overline{\mathbf{36}} + \overline{\mathbf{420}}\,, \end{aligned} \tag{6.1}$$

where the $\mathbf{28}$ representation in the first branching corresponds to the gauge potentials and the conjugate $\overline{\mathbf{28}}$ corresponds to the dual magnetic potentials, which cannot be included in the gauging. Recall that the embedding matrix $\Theta_M{}^\alpha$ living in the $\mathbf{912}$ representation encodes the coupling of the gauge fields to the $\mathrm{E}_{7(7)}$ generators (3.1), (4.9) and therefore its index M refers to the $\overline{\mathbf{28}}$ representation that is conjugate to the representation to which the gauge fields have been assigned. Note also that, according to (4.10), the T-tensor transforms in the conjugate representation (induced by the transformations of $\mathcal{V}$), as compared to the representation found for the embedding matrix $\Theta_M{}^\alpha$. The branchings of products of the relevant representations (6.1) that belong to the $\mathbf{912}$, and thus identify acceptable representations of a T-tensor, is conveniently summarized by the table below,

	$\mathbf{28}$	$\overline{\mathbf{28}}$
$\mathbf{63}$	$\mathbf{36} + \mathbf{420}$	$\overline{\mathbf{36}} + \overline{\mathbf{420}}$
$\mathbf{70}$	$\overline{\mathbf{420}}$	$\mathbf{420}$

(6.2)

Because only one $\mathbf{420}$ representation appears in the branching of the $\mathbf{912}$, the two $\mathbf{420}$ representations in (6.2) must coincide, and so must the $\overline{\mathbf{420}}$ representations. This implies that, if the embedding matrix $\Theta_M{}^\alpha$ had a contribution in the $\mathbf{420}$, it would describe a coupling of the gauge fields to the generators in the $\mathbf{70}$, but also, at the same time, induce a coupling of the dual gauge fields to the generators in the $\mathbf{63}$ representation of $\mathrm{SL}(8,\mathbb{R})$. Since this is not possible in a local field theory the embedding matrix must transform in the $\overline{\mathbf{36}}$ representation. Indeed, according to (6.2), the gauge group generators are then contained in the adjoint $\mathbf{63}$ representation of $\mathrm{SL}(8,\mathbb{R})$. Thus we conclude that all possible gaugings for the Lagrangian in the $\mathrm{SL}(8,\mathbb{R})$ basis are defined by an embedding matrix in the $\overline{\mathbf{36}}$ representation. The $\overline{\mathbf{36}}$ representation falls in different conjugacy classes with respect to $\mathrm{SL}(8,\mathbb{R})$, characterized by the eigenvalues of a symmetric 8×8 matrix θ_{AB}, which can be taken equal to ± 1 or 0. There are 44 nontrivial conjugacy classes, of which 24 correspond to inequivalent gaugings. For p eigenvalues $+1$, q eigenvalues -1 and r eigenvalues equal to 0, the resulting gauge group equals $\mathrm{CSO}(p,q,r)$. The matrix θ_{AB} is invariant under this group.

We stress that the embedding matrix completely determines the gauging. The embedding matrix $\Theta_{[AB]}{}^{C}{}_{D} \propto \delta^{C}_{[A}\,\theta_{B]D}$, where θ_{AB} denotes the component of the embedding matrix in the $\overline{\mathbf{36}}$ representation, defines gauge group generators in terms of the $\mathrm{SL}(8,\mathbb{R})$ generators $t_A{}^B$ (*cf.* (4.8))

$$t_{AB} = \Theta_{[AB]}{}^{C}{}_{D}\, t_C{}^D \propto \theta_{D[A}\, t_{B]}{}^D\,. \tag{6.3}$$

Indeed these are the generators corresponding to $\mathrm{CSO}(p,q,r)$. For instance, consider the case $p=8$, $q=r=0$. The embedding matrix is then invariant under the SO(8) subgroup of $\mathrm{SL}(8,\mathbb{R})$. There is just one SO(8) invariant contraction, which corresponds to the SO(8) gauging of [1]. In the case that $p+q=8$ and $r=0$, the embedding matrix is $\mathrm{SO}(p,q)$ invariant, and again, this identifies the latter as the unique gauge group. In the case that $r\neq 0$ we can use a contraction of $\mathrm{SO}(p,q)$, which leaves the Lagrangian, and in particular the T-tensor invariant. In this way we recover all the $\mathrm{CSO}(p,q,r)$ gaugings with $p+q+r=8$ found in [4] (*cf.* table 12 of [9]).

The $\mathrm{CSO}(p,q,r)$ gaugings with $r\neq 0$ can also be obtained from the $\mathrm{SO}(p+r,q)$ gauging by a contraction [4]. In order to show this let us briefly discuss the behaviour of the various quantities under the diagonal subgroup, *i.e.* the $\mathrm{SL}(8,\mathbb{R})$ matrices which are diagonal with eigenvalues λ_A (subject to the condition $\Pi_{A=1}^{8}\,\lambda_A = 1$). Under these transformations it follows from (2.17) and (2.21), that (no summation over repeated indices),

$$\begin{aligned} A_\mu^{AB} &\;\rightarrow\; \lambda_A\lambda_B\, A_\mu^{AB}\,,\\ u_{ij}{}^{AB} \pm v_{ijAB} &\;\rightarrow\; (\lambda_A\lambda_B)^{\pm 1}\,(u_{ij}{}^{AB} \pm v_{ijAB})\,. \end{aligned} \tag{6.4}$$

It is easy show that the Lagrangian (2.16) is indeed invariant under these transformations, but the terms induced by the gauging (*i.e.* both in the covariant derivative and in the T-tensor) are not. Decompose the 28 $\mathrm{SO}(p+r,q)$ generators as

$$\begin{pmatrix} \mathsf{A}_{p+q} & \mathsf{C}\eta \\ -\eta\mathsf{C}^{\mathrm{T}} & \mathsf{B}_r \end{pmatrix}\,, \tag{6.5}$$

where $\mathsf{A}_{p+q}^{\mathrm{T}} = -\eta\mathsf{A}\eta$ and $\mathsf{B}^{\mathrm{T}} = -\mathsf{B}$ with η the $\mathrm{SO}(p,q)$ invariant metric. Applying the transformation (6.4) has the effect of both rescaling the embedding matrix and the generators.

Consider a special transformation with $\lambda_A = \lambda$ for the first $p+q$ eigenvalues and $\lambda_A = \sigma$ for the last r eigenvalues, so that $\lambda^{p+q}\,\sigma^r = 1$. In that case A and B remain unchanged whereas one of the the off-diagonal blocks is multiplied by σ/λ and the other one by its inverse. The contraction of these generators with the gauge fields A_μ^{AB}, or the quantities $(u_{ij}{}^{AB} + v_{ijAB})$ as in the covariant derivative and the T-tensor, respectively, introduces additional factors λ^2, $\sigma\lambda$ or σ^2, depending on the index values. Performing now the singular limit $\sigma \to 0$ on finds that the contraction is proportional to λ^2 (which can be absorbed in the coupling constant, so that the covariant derivative and the T-tensor remain finite) times the generators (6.5) with B and one of the off-diagonal blocks suppressed. The dimension of the contracted gauge group $\mathrm{CSO}(p,q,r)$ is thus equal to $\frac{1}{2}(p+q)(p+q-1+2r)$, and the corresponding algebra consists of the generators A and $r(p+q)$ nilpotent generators residing in the off-diagonal block C.

6.2 Gauge groups embedded in the $\mathrm{E}_{6(6)}$ basis

The second class of gaugings concerns gauge groups that can be found in the $\mathrm{E}_{6(6)}$ basis defined in section 2. Again we start by giving the relevant branchings of the various $\mathrm{E}_{7(7)}$ representations with respect to $\mathrm{E}_{6(6)} \times \mathrm{O}(1,1)$. The branching of the **56**, **133** and **912** representations are given below (the subscript refers to the $\mathrm{SO}(1,1)$ weight),

$$\begin{aligned}
\mathbf{56} &= \overline{\mathbf{27}}_{-1} + \mathbf{1}_{-3} + \mathbf{27}_{+1} + \mathbf{1}_{+3}\,, \\
\mathbf{133} &= \mathbf{78}_0 + \overline{\mathbf{27}}_{+2} + \mathbf{27}_{-2} + \mathbf{1}_0\,, \\
\mathbf{912} &= \overline{\mathbf{351}}_{-1} + \mathbf{351}_{+1} + \overline{\mathbf{27}}_{-1} + \mathbf{27}_{+1} + \mathbf{78}_{-3} + \mathbf{78}_{+3}\,,
\end{aligned} \tag{6.6}$$

where in the second line $\mathbf{78}_0$ and $\mathbf{1}_0$ represent the adjoint representations of $\mathrm{E}_{6(6)}$ and of $\mathrm{O}(1,1)$, respectively. Just as before, we can conveniently summarize the branchings of $(\mathbf{56}\times\mathbf{133})\cap\mathbf{912}$ in a table,

	$\overline{\mathbf{27}}_{-1}$	$\mathbf{1}_{-3}$	$\mathbf{27}_{+1}$	$\mathbf{1}_{+3}$
$\mathbf{78}_0$	$\overline{\mathbf{351}}_{-1} + \overline{\mathbf{27}}_{-1}$	$\mathbf{78}_{-3}$	$\mathbf{351}_{+1} + \mathbf{27}_{+1}$	$\mathbf{78}_{+3}$
$\mathbf{27}_{-2}$	$\mathbf{78}_{-3}$		$\overline{\mathbf{351}}_{-1} + \overline{\mathbf{27}}_{-1}$	$\mathbf{27}_{+1}$
$\overline{\mathbf{27}}_{+2}$	$\mathbf{351}_{+1} + \mathbf{27}_{+1}$	$\overline{\mathbf{27}}_{-1}$	$\mathbf{78}_{+3}$	
$\mathbf{1}_0$	$\overline{\mathbf{27}}_{-1}$		$\mathbf{27}_{+1}$	

(6.7)

Again equivalent representations in this table must coincide, since the **912** contains just a single copy of each. With a similar reasoning as above, it follows that viable gaugings involve the gauge fields (in the $\overline{\mathbf{27}}_{-1} + \mathbf{1}_{-3}$ representation) coupling to $\mathrm{E}_{7(7)}$ generators belonging to the $\mathbf{78}_0 + \overline{\mathbf{27}}_{+2}$ representation and the corresponding embedding matrix is contained the $\mathbf{78}_{+3}$ representation (we recall that the embedding matrix is assigned to the representation that is conjugate with respect to one to which the gauge fields have been assigned).

This completely determines all possible gaugings in this basis. The gauge field in the $\mathbf{1}_{-3}$ representation couples to an element of the adjoint representation of $\mathrm{E}_{6(6)}$, whereas (part of) the gauge fields in the $\overline{\mathbf{27}}_{-1}$ representation couple to generators in the $\overline{\mathbf{27}}_{+2}$ representation of $\mathrm{E}_{6(6)} \times \mathrm{SO}(1,1)$. This gauging is not new and has an interpretation as a Scherk-Schwarz reduction from $d = 5$ maximal supergravity. Indeed, the $\mathrm{SO}(1,1)$ weights of the gauge fields are consistent with (5.3)): the graviphoton transforms in the $\mathbf{1}_{-3}$ representation and the 27 gauge fields from the five-dimensional theory transform in the $\overline{\mathbf{27}}_{-1}$ representation. In the Scherk-Schwarz reduction the graviphoton couples to one of the $\mathrm{E}_{6(6)}$ generators, whereas the remaining 27 gauge fields couple to generators in the $\overline{\mathbf{27}}_{+2}$ representation. This interpretation is also confirmed by the fact that the T-tensor transforms in the $\mathbf{78}_{-3}$ representation (remember that the representation is conjugate to the one for the embedding matrix), in accord with (5.7). The form of the gauge group generators follows from (2.26) and the dimension of the gauge group follows from the rank of the $\mathrm{E}_{6(6)}$ generator that has been selected. Hence we reproduce the result [7]. This particular Scherk-Schwarz reduction has been worked out in [23].

These above gaugings are thus defined by a family of embedding matrices which associate the graviphoton generator X_0 with one of the 78 generators t_0 in $\mathrm{E}_{6(6)}$, and the remaining gauge generators X_Λ with the nilpotent generators denoted by t_Λ in the $\overline{\mathbf{27}}_{+2}$ representation. From $[t_0, t_\Lambda] = M_\Lambda{}^\Sigma\, t_\Sigma$, it follows that the embedding matrix can then be represented as follows,

$$X_0 \;=\; t_0\,, \qquad X_\Lambda = M_\Lambda{}^\Sigma t_\Sigma\,, \tag{6.8}$$

so that the Lie algebra based on the generators $\{X_0,\, X_\Lambda\}$ takes the form,

$$[X_0,\, X_\Lambda] \;=\; M_\Lambda{}^\Sigma X_\Sigma\,, \qquad [X_\Lambda,\, X_\Sigma] = 0\,. \tag{6.9}$$

The null vectors of the matrix M correspond to gauge fields that do not participate in the gauging. As we discussed in section 5, when t_0 is a generator belonging to the compact subgroup of $\mathrm{E}_{6(6)}$, the corresponding potentials have stationary points with Minkowski ground states. This subgroup is equal to the group $\mathrm{USp}(8)$, which has conjugacy classes described in terms of four real parameters $m_1, \ldots, m_4$ (*cf.* [23]). Generically the matrix M has then three zero eigenvalues and 24 eigenvalues equal to the linear combinations $\pm m_i \pm m_j$ with $i > j$ taking the values $1, \ldots, 4$. These are the weights of the **27** representation of $\mathrm{USp}(8)$ which determine the mass matrix of the vector fields. The gauge group dimension takes odd values between 13 and 25. The 13-dimensional gauge group is equal to $\mathrm{CSO}(2,0,6)$.

6.3 Gauge groups embedded into $\mathrm{SU}^\star(8)$

As we discussed in section 2, ungauged Lagrangians exist with $\mathrm{SU}^\star(8)$ symmetry [8]. The group $\mathrm{SU}^\star(2n)$ is defined as the group of complex $2n \times 2n$ matrices U satisfying:

$$\mathsf{U}\mathsf{J} \;=\; \mathsf{J}\mathsf{U}^\star\,, \tag{6.10}$$

where J is the $\mathrm{Sp}(2n)$ invariant form. The compact subgroup is equal to $\mathrm{USp}(2n)$, and the real subgroup is equal to $\mathrm{SO}^\star(2n)$. Finally the groups $\mathrm{CSO}^\star(2p, 2n-2p)$ can be obtained by a contraction of $\mathrm{SO}^\star(2n)$.

In this basis $\mathrm{SU}^\star(8)$ acts block-diagonally on the gauge fields and their dual potentials. The relevant $\mathrm{E}_{7(7)}$ representations decompose with respect to $\mathrm{SU}^\star(8)$ precisely as in (6.1). However, in contradistinction to $\mathrm{SL}(8,\mathbb{R})$, the group $\mathrm{SU}^\star(8)$ corresponds to a real form of $\mathrm{SL}(8,\mathbb{C})$ with 36 generators defining the $\mathrm{USp}(8)$ subgroup and 27 generating the noncompact components of the group. The maximal compact subgroup can be defined by $\mathrm{USp}(8) = \mathrm{SU}^\star(8) \cap \mathrm{E}_{6(6)}$. The embedding of $\mathrm{SU}^\star(8)$ inside $\mathrm{E}_{7(7)}$ can be conveniently described by decomposing the adjoint of the latter with respect to $\mathrm{USp}(8)$:

$$\begin{aligned} \mathbf{133} &\;\to\; \mathbf{36} + \mathbf{42} + \mathbf{1} + \mathbf{27}_{\mathrm{nc}} + \mathbf{27}_{\mathrm{c}}\,, \\ \mathrm{Adj}(\mathrm{SU}^\star(8)) &\;\to\; \mathbf{36} + \mathbf{27}_{\mathrm{nc}}\,, \end{aligned} \tag{6.11}$$

where $\mathbf{27}_{\mathrm{nc}}$ denotes the noncompact linear combinations of the nilpotent generators $\mathbf{27}_{-2}$ and $\overline{\mathbf{27}}_{+2}$ from (6.6); $\mathbf{27}_{\mathrm{c}}$ denotes the corresponding compact combination. The $\mathbf{70}$ in the decomposition of the $\mathrm{Adj}(\mathrm{E}_{7(7)})$ representation with respect to $\mathrm{SU}^\star(8)$ consists therefore of the $\mathrm{USp}(8)$ representation $\mathbf{42}+\mathbf{1}+\mathbf{27}_{\mathrm{c}}$. The transformation from the $\mathrm{E}_{6(6)}$ basis to the $\mathrm{SU}^\star(8)$ basis was defined in (2.29).

The decomposition of the $\mathbf{912}$ with respect to $\mathrm{SU}^\star(8)$ takes the same form as for the $\mathrm{SL}(8,\mathbb{R})$ case. Therefore it follows that the embedding matrix must belong to the $\mathbf{36}$ representation of $\mathrm{SU}^\star(8)$. This representation has four nontrivial conjugacy classes, depending on the rank of the embedding matrix. For a nonsingular embedding matrix the gauge group equals the real subgroup $\mathrm{SO}^\star(8)$. Other gaugings correspond to the groups $\mathrm{CSO}^\star(2p, 8-2p)$ and can be described by an appropriate contraction, just as as described earlier for the $\mathrm{SL}(8,\mathbb{R})$ basis and analyzed in [8].

7 Some gaugings in $d = 5$ dimensions

As yet another application, we analyze some of the gaugings in $d = 5$ maximal supergravity. In this case there are no subtleties related to electric-magnetic dualities and the search for viable gauge groups should be based on arbitrary subgroups of $\mathrm{E}_{6(6)}$

without the need for referring to a specific basis. We consider several classes. First we assume that the gauge group is a subgroup of the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(6,\mathbb{R})$ maximal subgroup of $\mathrm{E}_{6(6)}$. Then we consider the case where the gauge group is embedded in the non-semisimple extension $\mathrm{SO}(5,5) \times \mathrm{SO}(1,1)$, which is also a maximal subgroup of $\mathrm{E}_{6(6)}$.

As proven in section 4 the embedding matrix for $d = 5$ dimensions must belong to the **351** representation of $\mathrm{E}_{6(6)}$, *i.e.*,

$$\mathbb{P}_{(351)\Lambda}{}^{a\,\Sigma}{}_{b}\,\Theta_\Sigma{}^b \;=\; \Theta_\Lambda{}^a\,, \tag{7.12}$$

where the projection operator was defined in (4.13). With respect to the $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(6,\mathbb{R})$ of $\mathrm{E}_{6(6)}$, the vector gauge fields, the $\mathrm{E}_{6(6)}$ generators and the embedding matrix decompose according to,

$$\begin{aligned}
\overline{\mathbf{27}} &\;\rightarrow\; (\mathbf{1},\overline{\mathbf{15}}) + (\mathbf{2},\mathbf{6})\,,\\
\mathbf{78} &\;\rightarrow\; (\mathbf{1},\mathbf{35}) + (\mathbf{3},\mathbf{1}) + (\mathbf{2},\mathbf{20})\,,\\
\mathbf{351} &\;\rightarrow\; (\mathbf{1},\overline{\mathbf{21}}) + (\mathbf{3},\mathbf{15}) + (\mathbf{2},\overline{\mathbf{84}}) + (\mathbf{2},\overline{\mathbf{6}}) + (\mathbf{1},\overline{\mathbf{105}})\,,
\end{aligned} \tag{7.13}$$

respectively. The table below summarizes how the embedding matrix couples the vector fields to the generators (we recall that the embedding matrix is assigned to the $(\mathbf{27}\times\mathbf{78})\cap\mathbf{351}$ representation),

	$(\mathbf{1},\mathbf{15})$	$(\mathbf{2},\overline{\mathbf{6}})$
$(\mathbf{1},\mathbf{35})$	$(\mathbf{1},\overline{\mathbf{21}}) + (\mathbf{1},\overline{\mathbf{105}})$	$(\mathbf{2},\overline{\mathbf{6}}) + (\mathbf{2},\overline{\mathbf{84}})$
$(\mathbf{3},\mathbf{1})$	$(\mathbf{3},\mathbf{15})$	$(\mathbf{2},\overline{\mathbf{6}})$
$(\mathbf{2},\mathbf{20})$	$(\mathbf{2},\overline{\mathbf{6}}) + (\mathbf{2},\overline{\mathbf{84}})$	$(\mathbf{3},\mathbf{15}) + (\mathbf{1},\overline{\mathbf{105}})$

(7.14)

where again equivalent representations in the table must be identified. Only the first two rows are, however, acceptable because those belong to the generators of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(6,\mathbb{R})$ and we assumed that the gauge group was embedded in this group. This leaves only one possible representation assignment for the embedding matrix, namely it should belong to the $(\mathbf{1},\overline{\mathbf{21}})$ representation and only the vector fields transforming in the $(\mathbf{1},\overline{\mathbf{15}})$ representation are involved in the gauging and couple to the generators in the adjoint representation of $\mathrm{SL}(6,\mathbb{R})$. The charged vector fields in the $(\mathbf{2},\mathbf{6})$ cannot participate in the gauging and must be dualized into antisymmetric tensor fields. The gaugings are again completely determined and correspond to the conjugacy classes of the $(\mathbf{1},\overline{\mathbf{21}})$ representation. They lead to the $\mathrm{CSO}(p,q,r)$ gauge groups with $p+q+r = 6$ found by [2, 5]. Apart from the conversion into tensor fields, this is entirely analogous to the discussion in $d = 4$ dimensions for the $\mathrm{SL}(8,\mathbb{R})$ basis.

A second application is based on $\mathrm{SO}(5,5) \times \mathrm{SO}(1,1)$. This semisimple group is not a maximal subgroup of $\mathrm{E}_{6(6)}$, but it becomes maximal upon including 16 additional nilpotent generators transforming in the $\overline{\mathbf{16}}_{+3}$ representation. We assume that the gauge group will be a subgroup of this non-semisimple maximal subgroup. The decompositions of the relevant $\mathrm{E}_{6(6)}$ representations with respect to the $\mathrm{SO}(5,5)\times\mathrm{SO}(1,1)$ subgroup is given by,

$$
\begin{aligned}
\overline{\mathbf{27}} &= \overline{\mathbf{16}}_{-1} + \mathbf{10}_{+2} + \mathbf{1}_{-4}\,,\\
\mathbf{78} &= \mathbf{45}_0 + \mathbf{1}_0 + \mathbf{16}_{-3} + \overline{\mathbf{16}}_{+3}\,,\\
\mathbf{351} &= \mathbf{144}_{+1} + \mathbf{16}_{+1} + \mathbf{45}_{+4} + \mathbf{120}_{-2} + \mathbf{10}_{-2} + \overline{\mathbf{16}}_{-5}\,.
\end{aligned}
\tag{7.15}
$$

The couplings induced by an embedding matrix solving (7.12) are shown below,

	$\mathbf{16}_{+1}$	$\mathbf{10}_{-2}$	$\mathbf{1}_{+4}$
$\mathbf{45}_0$	$\mathbf{144}_{+1}+\mathbf{16}_{+1}$	$\mathbf{10}_{-2}+\mathbf{120}_{-2}$	$\mathbf{45}_{+4}$
$\mathbf{1}_0$	$\mathbf{16}_{+1}$	$\mathbf{10}_{-2}$	
$\mathbf{16}_{-3}$	$\mathbf{120}_{-2}+\mathbf{10}_{-2}$	$\overline{\mathbf{16}}_{-5}$	$\mathbf{16}_{+1}$
$\overline{\mathbf{16}}_{+3}$	$\mathbf{45}_{+4}$	$\mathbf{144}_{+1}+\mathbf{16}_{+1}$	

(7.16)

Again equivalent representations for the embedding matrix should be identified as they appear with multiplicity one in the $\mathbf{351}$ representation. Furthermore, the generators belonging to the $\mathbf{16}_{-3}$ cannot be involved in the gauging, as they do not belong to the maximal subgroup that we have chosen. Therefore only two representations are allowed for the embedding matrix, namely the $\mathbf{144}_{+1}$ and the $\mathbf{45}_{+4}$ representation. As we will outline below, two particular classes corresponding to each of these representations can be immediately identified. No gaugings have been worked out so far with an embedding matrix that contains components from both representations. We will return to this elsewhere.

When the embedding matrix belongs to the $\mathbf{144}_{+1}$ representation, one can consider gauged supergravity in $d=6$ dimensions, whose embedding matrix must be in the $\mathbf{144}$ representation of the $\mathrm{SO}(5,5)$ duality group (*cf.* (4.6)). Upon dimensional reduction on S^1, one finds a T-tensor in the $\mathbf{144}_{-1}$ representation, which is indeed conjugate to the representation of the embedding matrix. However, not too much is known about gaugings for the $d=6$ theory (see, for example, [24])

When the embedding matrix belongs to the $\mathbf{45}_{+4}$ representation we are dealing with a Scherk-Schwarz reduction from $d=6$ dimensions, where ungauged maximal supergravity is invariant under $\mathrm{SO}(5,5)$ duality. To verify this, first consider the representations for the vector fields whose six-dimensional origin is as follows. The $\mathbf{1}_{-4}$ vector field corresponds to the graviphoton, the $\overline{\mathbf{16}}_{-1}$ vector fields originate from the 16 $d=6$ vector fields, and the $\mathbf{10}_{+2}$ vector fields originate from the 10 $d=6$ tensor fields. The $\mathrm{SO}(1,1)$ weights are in accord with the results given in (5.3). Regarding the tensors, we note that a tensor A_{MN} in 6 dimensions leads to a vector $A_{\mu 6}$ and a tensor

$A_{\mu\nu}$ in 5 dimensions, which, according to (5.3) have weights $+2$ and -2, respectively. However, a 2-rank tensor gauge field in 5 dimensions can be dualized into a vector field with opposite weight, so that all the vectors originating from tensor gauge fields in 6 dimensions carry the same SO(1,1) weight equal to $+2$. The T-tensor in this reduction must be in the $\mathbf{45}_{-4}$ representation, which is indeed conjugate to the representation of the embedding matrix identified above.

Just as in the four-dimensional case discussed in section 6.2, one identifies the representation $\mathbf{45}_{+4}$ with the complete family of five-dimensional gauge groups generated by $\{X_0,\, X_p\}_{p=1,\ldots,16}$, characterized by associating the generator X_0 which couples to the graviphoton in the $\mathbf{1}_{-4}$ representation with a generator t_0 of SO(5,5). The remaining generators X_p couple to the gauge fields in the $\overline{\mathbf{16}}_{-1}$ representation and are associated with the nilpotent generators t_p in the $\overline{\mathbf{16}}_{+3}$ representation. From $[t_0, t_p] = M_p{}^q\, t_q$ one finds the embedding matrix given by

$$X_0 \;=\; t_0\,, \qquad X_p = M_p{}^q\, t_q\,, \tag{7.17}$$

with the following gauge algebra generated by $\{X_0,\, X_p\}$,

$$[X_0,\, X_p] \;=\; M_p{}^q X_q\,, \qquad [X_p,\, X_q] = 0\,. \tag{7.18}$$

From (7.17) it follows that the null vectors of $M_p{}^q$ correspond to gauge fields that do not participate in the gauging and remain abelian. Obviously the maximal dimension of the gauge group is equal to 17. This means that at least 10 of the 27 vector fields of the $d = 5$ theory must be converted to charged tensor fields; at any rate these include the vectors in the $\mathbf{10}_{+2}$.

When X_0 belongs to the Lie algebra associated with the maximal compact subgroup of SO(5,5), which is equal to the rank-4 group H $=$ USp(4) $\times$ USp(4), it can be described by four real parameters m_1, m_2 and $\tilde{m}_1, \tilde{m}_2$, which denote the eigenvalues of X_0 in the $(\mathbf{4},\mathbf{1})$ and in the $(\mathbf{1},\mathbf{4})$ representation, respectively. As discussed in section 5 the corresponding potential (which is nonnegative) has minima with zero cosmological constant. For generic values of the four parameters we can determine the physical masses, collected in table 5.

With respect to H $=$ USp(4) $\times$ USp(4) the gravitini transform in the $(\mathbf{4},\mathbf{1}) + (\mathbf{1},\mathbf{4})$ and the spinors in the $(\mathbf{4},\mathbf{5}) + (\mathbf{5},\mathbf{4}) + (\mathbf{4},\mathbf{1}) + (\mathbf{1},\mathbf{4})$ representations, respectively. The spinors corresponding to the $(\mathbf{4},\mathbf{1}) + (\mathbf{1},\mathbf{4})$ representation are absorbed into the massive gravitini through a super-Brout-Englert-Higgs mechanism. The masses of the gravitini are given by the eigenvalues of the matrix describing the action of X_0 on this representation. The 16 vector fields and the 10 tensor fields transform under H in the $(\mathbf{4},\mathbf{4})$ and in the $(\mathbf{5},\mathbf{1}) + (\mathbf{1},\mathbf{5})$ representations, respectively. We observe that there are two neutral, massless tensor fields which can be consistently dualized back into vector

field	spin/helicity	H-rep	masses	#	dof's
graviton	5	$(\mathbf{1},\mathbf{1})$	0	1	5
gravitini	$(3,2)$	$(\mathbf{4},\mathbf{1})$	$\lvert m_r\rvert$	2	24
	$(2,3)$	$(\mathbf{1},\mathbf{4})$	$\lvert\tilde{m}_r\rvert$	2	24
vectors	$(2,2)$	$(\mathbf{4},\mathbf{4})$	$\lvert m_r \pm \tilde{m}_s\rvert$	2	64
	3	$(\mathbf{1},\mathbf{1})$	0	1	3
tensors	$(3,1)$	$(\mathbf{5},\mathbf{1})$	$\begin{cases} \lvert m_1 \pm m_2\rvert \\ 0 \end{cases}$	2 1	12 3
	$(1,3)$	$(\mathbf{1},\mathbf{5})$	$\begin{cases} \lvert\tilde{m}_1 \pm \tilde{m}_2\rvert \\ 0 \end{cases}$	2 1	12 3
spinors	$(1,2)$	$(\mathbf{4},\mathbf{5})$	$\begin{cases} \lvert m_r \pm \tilde{m}_1 \pm \tilde{m}_2\rvert \\ \lvert m_r\rvert \end{cases}$	2 2	32 8
	$(2,1)$	$(\mathbf{5},\mathbf{4})$	$\begin{cases} \lvert m_1 \pm m_2 \pm \tilde{m}_s\rvert \\ \lvert\tilde{m}_s\rvert \end{cases}$	2 2	32 8
scalars	0	$(\mathbf{1},\mathbf{1})$	0	1	1
		$(\mathbf{5},\mathbf{5})$	$\begin{cases} \lvert m_1 \pm m_2 \pm \tilde{m}_1 \pm \tilde{m}_2\rvert \\ \lvert m_1 \pm m_2\rvert \\ \lvert\tilde{m}_1 \pm \tilde{m}_2\rvert \\ 0 \end{cases}$	2 2 2 1	16 4 4 1

Table 5: Mass spectrum of maximal $d = 5$ gauged supergravity with an embedding matrix in the $(\mathbf{10},\mathbf{1})+(\mathbf{1},\mathbf{10})$ representation of $\mathrm{H} = \mathrm{USp}(4)\times\mathrm{USp}(4)$. Massless states transform under SU(2) helicity rotations; the dimension of the corresponding representation is indicated by a single number. Massive states transform under $\mathrm{SU}(2)\times\mathrm{SU}(2)$ spatial rotations, so that their spin-content is characterized by two numbers.

fields. The scalar fields transform under H in the $(\mathbf{1},\mathbf{1})+(\mathbf{4},\mathbf{4})+(\mathbf{5},\mathbf{5})$ representation. The scalars in the $(\mathbf{4},\mathbf{4})$ arise from the internal components of the vector fields in six dimensions. They are absorbed into the massive gauge vectors.

We would like to conclude this discussion with a remark on the four-dimensional models obtained by simple dimensional reduction on a circle from five-dimensional gauged maximal supergravity. As pointed out in section 6.2, dimensional reduction of the ungauged five-dimensional theory leads to the 28 four-dimensional vector fields described in (2.27). On the other hand, the $\mathrm{CSO}(p,q,r)$ gaugings in five dimensions break the $\mathrm{E}_{6(6)}$ invariance, not only as a consequence of the minimal couplings, but also because the vector fields in the $(\mathbf{2},\mathbf{6})$ have to be dualized into tensor fields in the $(\mathbf{2},\overline{\mathbf{6}})$. As a consequence of this dualization, dimensional reduction of the $\mathrm{CSO}(p,q,r)$ gauged theories gives rise to four-dimensional vector fields transforming as in (2.28), *i.e.*, it leads to four-dimensional gaugings in the $\mathrm{SL}(2,\mathbb{R})\times\mathrm{SO}(1,1)\times\mathrm{SL}(6,\mathbb{R})$ basis. Since the graviphoton is ungauged, it may still be dualized: $(\mathbf{1},\mathbf{1})_{-3}\rightarrow(\mathbf{1},\mathbf{1})_{+3}$. This yields a gauged model in the $\mathrm{SL}(8,\mathbb{R})$ basis, *cf.* (2.24), with gauge group $\mathrm{CSO}(p,q,r+2)$ that has been considered in section 6.1. This has also been discussed in [8].

Acknowledgement

We thank H. Nicolai and S. Ferrara for useful discussions. This work is partly supported by EU contracts HPRN-CT-2000-00122 and HPRN-CT-2000-00131. M.T. is supported by a European Community Marie Curie Fellowship under contract HPMF-CT-2001-01276.

A Projectors on the embedding matrix: a general discussion

In this appendix, we explicitly construct the projectors onto the irreducible representations in the tensor product of the fundamental with the adjoint representation of an arbitrary simple group G. These projectors for G $=\mathrm{E}_{7(7)}$, and G $=\mathrm{E}_{6(6)}$ have been used in the main text to correctly identify the embedding matrices in $d=4$ and $d=5$, respectively.

Let us assume that the product of a fundamental representation $\mathbf{D}(\mathbf{\Lambda})$ times the adjoint decomposes in the direct sum of $\mathbf{D}(\mathbf{\Lambda})$ plus two other representations, $\mathbf{D_1}$ and $\mathbf{D_2}$,

$$\mathbf{D}(\mathbf{\Lambda})\times\mathbf{Adj}(\mathbf{G})\rightarrow\mathbf{D}(\mathbf{\Lambda})+\mathbf{D_1}+\mathbf{D_2}\;. \tag{A.1}$$

As far as the lowest-dimensional fundamental representation in concerned, the above branching rule holds true for any simple group with the exception of E_8 for which the fundamental coincides with the adjoint representation. The branching also holds for orthogonal groups when the fundamental representation is replaced by the spinor representation . Denote $d_\Lambda = \dim(\mathbf{D}(\mathbf{\Lambda}))$, $d = \dim(\mathrm{G})$, and $\{t^\alpha\}$ $(\alpha = 1, \ldots, d)$ the generators of G in the $\mathbf{D}(\mathbf{\Lambda})$ representation. Furthermore, let C_θ, C_Λ be the Casimirs of the adjoint and fundamental representations, respectively. We define the invariant matrix $\eta^{\alpha\beta} = \mathrm{Tr}(t^\alpha t^\beta)$ and use it to rise and lower the adjoint indices; it is related to the Cartan-Killing metric $\kappa^{\alpha\beta}$ by

$$\kappa^{\alpha\beta} \quad = \quad \frac{d}{C_\Lambda d_\Lambda}\,\eta^{\alpha\beta}\ . \tag{A.2}$$

Using the definition of the Casimir operator, $C_\Lambda\,\mathbb{1}_{d_\Lambda} = \kappa_{\alpha\beta}t^\alpha t^\beta$, we have the following relation

$$f_{\alpha\beta}{}^{\gamma}\, f^{\alpha\beta}{}_{\sigma} \quad = \quad -\frac{d}{d_\Lambda}\, C_{\mathrm{r}}\, \delta^\gamma_\sigma\ , \qquad \text{with} \quad C_{\mathrm{r}} = \frac{C_\theta}{C_\Lambda} = \frac{d_\Lambda}{d}\frac{g^\vee}{\tilde{I}_\Lambda}\ , \tag{A.3}$$

where $g^\vee$ is the dual Coxeter number and $\tilde{I}_\Lambda$ is the Dynkin index of the fundamental representation. In the simply laced case there is a useful formula:

$$C_{\mathrm{r}} \quad = \quad \frac{d_\Lambda}{d}\left(\frac{d}{r} - 1\right)\frac{1}{\tilde{I}_\Lambda}\ , \tag{A.4}$$

with r the rank of G.

Denote the projectors on the representations in (A.1) by $\mathbb{P}_{\mathbf{D}(\mathbf{\Lambda})}$, $\mathbb{P}_{\mathbf{D_1}}$, $\mathbb{P}_{\mathbf{D_2}}$ which sum to the identity on $\mathbf{D}(\mathbf{\Lambda}) \times \mathbf{Adj}(\mathbf{G})$. These three projectors can be expressed in terms of three independent objects, namely:

$$\begin{aligned}
\mathbb{P}_{\mathbf{D}(\mathbf{\Lambda})\mathbf{M}}{}^{\alpha N}{}_{\beta} &= \frac{d_\Lambda}{d}\,(t^\alpha t_\beta)_M{}^N\ ,\\
\mathbb{P}_{\mathbf{D_1}M}{}^{\alpha N}{}_{\beta} &= a_1\,\delta^\alpha{}_\beta\,\delta_M{}^N + a_2\,(t_\beta t^\alpha)_M{}^N + a_3\,(t^\alpha t_\beta)_M{}^N\ ,\\
\mathbb{P}_{\mathbf{D_2}M}{}^{\alpha N}{}_{\beta} &= (1-a_1)\,\delta^\alpha{}_\beta\,\delta_M{}^N - a_2\,(t_\beta t^\alpha)_M{}^N - (d_\Lambda/d + a_3)\,(t^\alpha t_\beta)_M{}^N\ ,
\end{aligned} \tag{A.5}$$

with constants a_1, a_2, a_3. Making use of the fact that only three representations appear in the decomposition (A.1), these coefficients may be determined by computing the contractions of various products of the projectors (A.5). This yield

$$\begin{aligned}
a_1 &= \frac{d_\Lambda\,(4 + (C_{\mathrm{r}} - 4)d)) + \Delta\,((C_{\mathrm{r}} - 2)d - 2))}{(10 + d(C_{\mathrm{r}} - 8) + d^2(C_{\mathrm{r}} - 2))\,d_\Lambda}\ ,\\
a_2 &= -\frac{2\,(4 + (C_{\mathrm{r}} - 4)d))\,((d-1)d_\Lambda - 2\Delta)}{(10 + d(C_{\mathrm{r}} - 8) + d^2(C_{\mathrm{r}} - 2))\,C_{\mathrm{r}} d}\ ,\\
a_3 &= \frac{-d_\Lambda\,(4 + (C_{\mathrm{r}} - 4)d))\,(2 + (C_{\mathrm{r}} - 2)d) + \Delta\,(16(d-1) - 10(d-1)C_{\mathrm{r}} + C_{\mathrm{r}}^2 d)}{(10 + d(C_{\mathrm{r}} - 8) + d^2(C_{\mathrm{r}} - 2))\,C_{\mathrm{r}} d}\ ,
\end{aligned}$$

G	$g^\vee$	d_Λ	$\tilde{I}_\Lambda$	Δ	a_1	a_2	a_3
$\mathrm{A_r}$	$r+1$	$r+1$	$\frac{1}{2}$	$\frac{1}{2}(r-1)(r+1)(r+2)$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2r}$
$\mathrm{B_r}$	$2r-1$	$2r+1$	1	$\frac{1}{3}r(4r^2-1)$	$\frac{1}{3}$	$-\frac{2}{3}$	0
$\mathrm{B_r}$	$2r-1$	2^r	2^{r-3}	$2^{r+1}\,r$	$\frac{2}{2r-1}$	$-2^{r-1}\frac{1}{2r-1}$	$2^{r-1}\,\frac{2r-7}{4r^2-1}$
$\mathrm{C_r}$	$r+1$	$2r$	$\frac{1}{2}$	$\frac{8}{3}r(r^2-1)$	$\frac{2}{3}$	$-\frac{2}{3}$	$-\frac{2}{1+2r}$
$\mathrm{D_r}$	$2r-2$	$2r$	1	$\frac{2}{3}r(2r^2-3r+1)$	$\frac{1}{3}$	$-\frac{2}{3}$	0
$\mathrm{D_r}$	$2r-2$	2^{r-1}	2^{r-4}	$2^{r-1}\,(2r-1)$	$\frac{1}{r-1}$	$-2^{r-3}\frac{1}{r-1}$	$2^{r-3}\frac{(r-4)}{r\,(r-1)}$
$\mathrm{G_2}$	4	7	1	27	$\frac{3}{7}$	$-\frac{6}{7}$	$-\frac{3}{14}$
$\mathrm{F_4}$	9	26	3	273	$\frac{1}{4}$	$-\frac{3}{2}$	$\frac{1}{4}$
$\mathrm{E_6}$	12	27	3	351	$\frac{1}{5}$	$-\frac{6}{5}$	$\frac{3}{10}$
$\mathrm{E_7}$	18	56	6	912	$\frac{1}{7}$	$-\frac{12}{7}$	$\frac{4}{7}$

Table 6: Coefficients for the projector $\mathbb{P}_{\mathbf{D_1}}$ for the various algebras.

with $\Delta = \dim(\mathbf{D_1})$. Moreover, Δ is determined to be

$$\Delta \;=\; \frac{d_\Lambda}{2}\left[d-1+\frac{\sqrt{C_\mathrm{r}}\,(10+d(C_\mathrm{r}-8)+d^2(C_\mathrm{r}-2))}{\sqrt{256(d-1)+C_\mathrm{r}(100+4d(5C_\mathrm{r}-38)+(C_\mathrm{r}-2)^2d^2)}}\right]\,. \tag{A.6}$$

In table 6 the relevant data are collected for all simple Lie algebras except $\mathrm{E_8}$ (for which the relevant projectors have been computed in [25]). In particular, equation (A.6) correctly reproduces the dimensions $\Delta = 912$, $\Delta = 351$ for G $=\mathrm{E_{7(7)}}$, and G $=\mathrm{E_{6(6)}}$, respectively. Moreover, the relevant projectors (4.12), (4.13) are obtained from (A.5).

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